CONTINUITY PROPERTIES OF SCHRÖDINGER SEMIGROUPS WITH MAGNETIC FIELDS

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The objects of the present study are one-parameter semigroups generated by Schrödinger operators with fairly general electromagnetic potentials. More precisely, we allow scalar potentials from the Kato class and impose on the vector potentials only local Kato-like conditions. The configuration space is supposed to be an arbitrary open subset of multi-dimensional Euclidean space; in case that it is a proper subset, the Schrödinger operator is rendered symmetric by imposing Dirichlet boundary conditions. We discuss the continuity of the image functions of the semigroup and show local-norm-continuity of the semigroup in the potentials. Finally, we prove that the semigroup has a continuous integral kernel given by a Brownian-bridge expectation. Altogether, the article is meant to extend some of the results in B. Simon's landmark paper [Bull. Amer. Math. Soc. (N.S.) **7**, 447–526 (1982)] to non-zero vector potentials and more general configuration spaces.

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Contents

1   Introduction

In non-relativistic quantum physics Schrödinger operators with magnetic fields play an important rôle [26, 27]. In suitable physical units they are given by differential expressions of the form

$$H_\Lambda(A, V) = \frac{1}{2}\left(-i\nabla - A(x)\right)^2 + V(x) \tag{1.1}$$

over the configuration space $\Lambda \subseteq \mathbb{R}^d$. Here the scalar potential V is a real-valued function on Λ representing the potential energy. The vector potential A is an $\mathbb{R}^d$-valued function on Λ giving rise to the magnetic field $\nabla \times A$. For the purposes of quantum physics $H_\Lambda(A, V)$ has to be given a precise meaning as a self-adjoint operator acting on the Hilbert space $\mathrm{L}^2(\Lambda)$ of complex-valued wave-functions on Λ. Clearly, if $\Lambda \neq \mathbb{R}^d$ this requires to impose boundary conditions on the wave functions in the domain (of definition) of $H_\Lambda(A, V)$.

As impressively demonstrated in Simon's landmark paper [58], many spectral properties of $H_\Lambda(A, V)$ can efficiently be derived by studying the Schrödinger semigroup $\left\{\mathrm{e}^{-tH_\Lambda(A,V)}\right\}_{t\geq 0}$. The latter can be done by probabilistic techniques via the Feynman-Kac-Itô path-integral formula. In fact, the works [10, 58] make essential use of this approach for the case $A = 0$ and $\Lambda = \mathbb{R}^d$, that is, when both the magnetic field vanishes and the configuration space is the whole Euclidean space. For $A = 0$ and $\Lambda \subseteq \mathbb{R}^d$ some results may be found in [64, Chapter 1]. Another recent monograph further elucidating the relation between "quantum potential theory" and Feynman-Kac processes is [11].

The main goal of the present work is to extend some of the key results in [10, 58] on continuity properties to rather general A and Λ. In doing so, we follow [58] in restricting

ourselves to Kato decomposable [37] scalar potentials V. The class of vector potentials A considered will only be restricted by local Kato-like conditions. The configuration space will either be $\Lambda = \mathbb{R}^d$ or an arbitrary non-empty open subset $\Lambda \subset \mathbb{R}^d$. Since vector potentials in one spatial dimension are of no physical interest, we will only consider dimensions $d \geq 2$. As for the necessary boundary conditions for $\Lambda \neq \mathbb{R}^d$, we are only able to handle Dirichlet conditions. Apart from that, in our opinion, the resulting class of Schrödinger operators with magnetic fields is general enough to cover most systems of physical interest with finite ground-state energy. Continuity properties of the generated semigroups have turned out to be valuable in obtaining interesting results in that field of mathematical physics where magnetic fields play a major rôle. We mention so-called magnetic Lieb-Thirring inequalities [22], heat-kernel estimates [21, 23] and the existence and Lifshits tailing of the integrated density of states of random Schrödinger operators [8, 67, 9, 24]. In proving the statements of the present paper we closely follow [10, 58, 64] using the probabilistic approach. Since we aim at a reasonably self-contained presentation which is intelligible for many readers, our arguments are perhaps more detailed than usual.

The reader should note that Simon's work [58] is followed by several other interesting developments in semigroup theory relating to Schrödinger operators. While some of them rely on probabilistic techniques, others do not. In a setting closest to that of the present paper one of us [34, 35] has considerably relaxed the conditions imposed on the vector potential A. Related results with $A \neq 0$ but with weaker assumptions on the scalar potential V have been obtained in [42] building on the works [68, 39] for $A = 0$. Basically, in [68, 39, 42] the negative part of V no longer needs to be infinitesimally form-bounded relative to the unperturbed Schrödinger operator $H_\Lambda(0,0)$. There has also been a lot of activity [14, 15, 16, 43] in studying perturbations of positivity preserving semigroups with generators more general than $H_\Lambda(0,0)$. Extensions into other directions investigate perturbations by measures instead of functions [7, 3, 28, 29, 63, 61]. Last but not least we mention the progress [62] for Schrödinger-like semigroups with an underlying configuration space which is only locally Euclidean, see also [15, 16].

For a wealth of information on regularity and spectral properties of Schrödinger operators (for $\Lambda = \mathbb{R}^d$) with or without magnetic field we recommend the recent works [32, 31, 45], see also [44] and references therein. In contrast to the present paper these works do not rely on semigroups.

2   Basic definitions and representations

In this section we fix our basic notation and give a short compilation of the classes of scalar potentials V and vector potentials A which we will consider. Moreover, we will give a precise definition of the Schrödinger operator (1.1) and recall the appropriate Feynman-Kac-Itô formula for its semigroup.

The open ball of radius $\varrho > 0$ centered about the origin in the ν-dimensional Euclidean space $\mathbb{R}^\nu$, $\nu \in \mathbb{N}$, is denoted by

$$B_\varrho := \{x \in \mathbb{R}^\nu : |x| < \varrho\}. \tag{2.1}$$

Here $|x| := (x \cdot x)^{1/2}$ is the norm of $x = (x_1, \ldots, x_\nu) \in \mathbb{R}^\nu$ derived from the scalar product

$x \cdot y := \sum_{j=1}^{\nu} x_j y_j$. Given a subset Ω of $\mathbb{R}^\nu$ we write

$$\omega \mapsto \chi_\Omega(\omega) := \begin{cases} 1 & \text{for } \omega \in \Omega \\ 0 & \text{otherwise} \end{cases} \tag{2.2}$$

for its indicator function. The closure of Ω is written as $\overline{\Omega}$ and its boundary is denoted by $\partial\Omega$.

We denote the nabla operator $\left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_\nu}\right)$ in $\mathbb{R}^\nu$ as ∇ and the Lebesgue measure on the Borel subsets of $\mathbb{R}^\nu$ as $\mathrm{d}x$. All real-valued or complex-valued functions on $\mathbb{R}^\nu$ are assumed to be Borel measurable. If not stated otherwise, we identify functions which differ only on sets of Lebesgue measure zero.

The Banach space $\mathrm{L}^p(\Omega)$ of pth-power Lebesgue-integrable functions ($1 \leq p \leq \infty$) on a Borel set $\Omega \subseteq \mathbb{R}^\nu$ of non-zero Lebesgue measure consists of complex-valued functions $\psi : \Omega \to \mathbb{C}$ such that the norm

$$\|\psi\|_p := \begin{cases} \left(\int_\Omega |\psi(x)|^p \, \mathrm{d}x\right)^{1/p} & p < \infty \\ \operatorname*{ess\,sup}_{x \in \Omega} |\psi(x)| & \text{for} \quad p = \infty \end{cases} \tag{2.3}$$

is finite. We recall that $\mathrm{L}^2(\Omega)$ becomes a Hilbert space when equipped with the scalar product

$$\langle \phi | \psi \rangle := \int_\Omega (\phi(x))^* \, \psi(x) \, \mathrm{d}x. \tag{2.4}$$

Here the star denotes complex conjugation. The norm of an operator $X : \mathrm{L}^p(\Omega) \to \mathrm{L}^q(\Omega)$, $1 \leq p, q \leq \infty$ is defined as

$$\|X\|_{p,q} := \sup_{\|\psi\|_p = 1} \|X\psi\|_q. \tag{2.5}$$

The space $\mathrm{L}^p_{\text{unif, loc}}(\Omega)$ of uniformly locally p^{th}-power integrable functions ($1 \leq p \leq \infty$) consists of functions $\psi : \Omega \to \mathbb{C}$ such that the norm

$$\|\psi\|_{\mathrm{L}^p_{\text{unif, loc}}(\Omega)} := \begin{cases} \sup_{x \in \mathbb{R}^\nu} \left(\int_\Omega |\psi(y)|^p \chi_{B_1}(x - y) \, \mathrm{d}y\right)^{1/p} & p < \infty \\ \operatorname*{ess\,sup}_{x \in \Omega} |\psi(x)| & \text{for} \quad p = \infty \end{cases} \tag{2.6}$$

is finite. The space $\mathrm{L}^p_{\text{loc}}(\Omega)$ of locally p^{th}-power integrable functions ($1 \leq p \leq \infty$) consists of functions $\psi : \Omega \to \mathbb{C}$ such that $\psi\chi_K \in \mathrm{L}^p(\Omega)$ for all compact $K \subseteq \Omega$.

The positive part f^+ and the negative part f^- of a real-valued function f on Ω are defined by

$$f^\pm(x) := \sup\{\pm f(x), 0\}, \quad x \in \Omega. \tag{2.7}$$

In case that $\Omega \subseteq \mathbb{R}^\nu$ is open, we write $\mathcal{C}(\Omega)$ for the space of complex-valued continuous functions on Ω. The subspace of arbitrarily often differentiable functions with compact support inside Ω is written as $\mathcal{C}_0^\infty(\Omega)$.

In the following Λ denotes a fixed, non-empty, open, not necessarily proper subset of $\mathbb{R}^d$, $d \geq 2$. We stress that we will always assume $d \geq 2$. In physical terms, Λ serves as the **configuration space**. Any complex-valued function f defined *a priori* on Λ will be understood to be trivially extended to $\mathbb{R}^d$ by defining $f(x) := 0$ for $x \notin \Lambda$ without further notice. We use this **extension convention** to injectively embed $L^p(\Lambda)$ into $L^p(\mathbb{R}^d)$, $1 \leq p \leq \infty$, and thus have $L^p(\Lambda) \subseteq L^p(\mathbb{R}^d)$ and similarly $\mathcal{C}_0^\infty(\Lambda) \subseteq \mathcal{C}_0^\infty(\mathbb{R}^d)$ etc. We write

$$g_\varrho(x) := \chi_{B_\varrho}(x) \begin{cases} -\ln|x| & \\ |x|^{2-d} & \end{cases} \text{ for } \begin{matrix} d = 2 \\ d \geq 3 \end{matrix} \tag{2.8}$$

for the repulsive Newton-Coulomb potential on $\mathbb{R}^d$ truncated outside the Ball B_ϱ.

Definition 2.1

> *A* **scalar potential** *is a real-valued function V on Λ. The function V is said to be in the* **Kato class** $\mathcal{K}(\mathbb{R}^d)$ *if*
>
> $$\lim_{\varrho \downarrow 0} \sup_{x \in \mathbb{R}^d} \int g_\varrho(x - y) \, |V(y)| \, \mathrm{d}y = 0. \tag{2.9}$$
>
> *The function V is said to be in the* **local Kato class** $\mathcal{K}_{loc}(\mathbb{R}^d)$ *if $V\chi_K \in \mathcal{K}(\mathbb{R}^d)$ for all compact $K \subset \mathbb{R}^d$. The function V is said to be* **Kato decomposable**, *in symbols $V \in \mathcal{K}_\pm(\mathbb{R}^d)$, if $V^+ \in \mathcal{K}_{loc}(\mathbb{R}^d)$ and $V^- \in \mathcal{K}(\mathbb{R}^d)$. The* **Kato norm** *is given by*
>
> $$\|V\|_{\mathcal{K}(\mathbb{R}^d)} := \sup_{x \in \mathbb{R}^d} \int g_1(x - y) \, |V(y)| \, \mathrm{d}y. \tag{2.10}$$

Note that due to the above extension convention it is reasonable to state $V \in \mathcal{K}(\mathbb{R}^d)$ even if V is *a priori* only defined on $\Lambda \subseteq \mathbb{R}^d$.

The following remarks are borrowed from [13, Chapter 1.2], [2, Section 4] and [11, Chapter 3].

Remarks 2.2

> *i*) To find out whether or not a function belongs to the Kato class or local Kato class the following inclusions may be helpful. For $p > \frac{d}{2}$ one has
>
> $$L^p_{\text{unif, loc}}(\mathbb{R}^d) \subset \mathcal{K}(\mathbb{R}^d) \subset L^1_{\text{unif, loc}}(\mathbb{R}^d), \tag{2.11}$$
> $$L^p_{\text{loc}}(\mathbb{R}^d) \subset \mathcal{K}_{\text{loc}}(\mathbb{R}^d) \subset L^1_{\text{loc}}(\mathbb{R}^d). \tag{2.12}$$

> *ii*) The Kato class is complete with respect to the Kato norm [58, Erratum], [68, Section 5], that is, $\mathcal{K}(\mathbb{R}^d)$ is a Banach space. The inclusions (2.11) hold also with respect to convergence in norm, that is, convergence in $\| \bullet \|_{L^p_{\text{unif, loc}}(\mathbb{R}^d)}$ implies convergence in $\| \bullet \|_{\mathcal{K}(\mathbb{R}^d)}$ which in turn implies convergence in $\| \bullet \|_{L^1_{\text{unif, loc}}(\mathbb{R}^d)}$, where $p > \frac{d}{2}$.

iii) Let $2H_\Lambda(0,0)$ denote the self-adjoint Friedrichs-extension on the Hilbert space $L^2(\Lambda)$ of the negative Laplacian $-\nabla \cdot \nabla$ on $\mathcal{C}_0^\infty(\Lambda)$, that is, $-2H_\Lambda(0,0)$ is the usual Laplacian on Λ with Dirichlet boundary conditions. Then any $V \in \mathcal{K}(\mathbb{R}^d)$ is infinitesimally form-bounded [50, Definition p. 168] relative to $H_\Lambda(0,0)$ [11, Corollary to Theorem 3.25]. Alternatively, this follows from [37, Proposition 2.1] or [2, Theorem 4.7] used with the fact that, by the extension convention, the form domain of $H_\Lambda(0,0)$ is a subset of the form domain of $H_{\mathbb{R}^d}(0,0)$. The class $\mathcal{K}(\mathbb{R}^d)$ is nearly maximal with respect to infinitesimal form-boundedness [13, Theorem 1.12]. Consequently, the set of Kato-decomposable functions should contain all physically relevant scalar potentials, which lead to Schrödinger operators (1.1) with a finite ground-state energy.

iv) An example illustrating the admissible local singularities of scalar potentials in $\mathcal{K}(\mathbb{R}^d)$ for $d \geq 3$ is

$$V_\mu(x) := \frac{\Theta_{1/2}(x)}{|x|^2 \, |\ln |x||^\mu}, \quad \mu > 0, \tag{2.13}$$

where $\Theta_{1/2}$ is some real-valued function in $\mathcal{C}_0^\infty(\mathbb{R}^d)$ which vanishes outside the ball $B_{3/4}$ and furthermore obeys $\Theta_{1/2}(x) = 1$ for $|x| < 1/2$. According to [13, Example (a) p.8], [2, Proposition 4.10] one has the equivalence

$$V_\mu \in \mathcal{K}(\mathbb{R}^d) \Leftrightarrow \mu > 1. \tag{2.14}$$

Note that, for $d = 3$, V_μ obeys the Rollnik condition [53, Chapter I] whenever $\mu > \frac{1}{2}$, see [53, Example I.6.3]. Finally, V_μ is infinitesimally form-bounded relative to $H_{\mathbb{R}^d}(0,0)$ for all $\mu > 0$.

Useful for handling Kato-decomposable functions is the following possibility to approximate them by nice functions.

Proposition 2.3

Let $V \in \mathcal{K}_\pm(\mathbb{R}^d)$. Then there is a sequence $\{V_n\}_{n\in\mathbb{N}} \subset \mathcal{C}_0^\infty(\mathbb{R}^d)$ such that

$$\lim_{n\to\infty} \left\| (V - V_n) \, \chi_K \right\|_{\mathcal{K}(\mathbb{R}^d)} = 0 \tag{2.15}$$

for all compact $K \subset \mathbb{R}^d$ and

$$\sup_{n\in\mathbb{N}} \sup_{x\in\mathbb{R}^d} \int g_\varrho(x - y) \, V_n^-(y) \, \mathrm{d}y \leq \sup_{x\in\mathbb{R}^d} \int g_\varrho(x - y) \, V^-(y) \, \mathrm{d}y \tag{2.16}$$

for all $0 < \varrho \leq 1$.

This approximability and the idea of its proof have been stated in [58, § B.10]. The approximating sequence $\{V_n\}$ may be constructed from V in the following standard way. The potential V is smoothly truncated outside an increasingly large ball and then mollified by convolution with an approximate delta function in $\mathcal{C}_0^\infty(\mathbb{R}^d)$. Since $V \in \mathcal{K}_\pm(\mathbb{R}^d)$ is locally integrable, the resulting approximations are in $\mathcal{C}_0^\infty(\mathbb{R}^d)$. Then patiently estimating shows that (2.15) and (2.16) can be fulfilled. The details can be found in Appendix A.

We now define the classes of vector potentials to be dealt with in the sequel.

Definition 2.4

A **vector potential** *is an $\mathbb{R}^d$-valued function A on Λ. A vector potential A is said to be in the class $\mathcal{H}(\mathbb{R}^d)$, if its squared norm $A^2 := A \cdot A$ and its divergence $\nabla \cdot A$, considered as a distribution on $\mathcal{C}_0^\infty(\mathbb{R}^d)$, are both in $\mathcal{K}(\mathbb{R}^d)$. It is said to be in the class $\mathcal{H}_{loc}(\mathbb{R}^d)$, if both A^2 and $\nabla \cdot A$ are in $\mathcal{K}_{loc}(\mathbb{R}^d)$.*

Remarks 2.5

i) As it should be from the physical point of view, only local regularity conditions are required for a vector potential to lie in $\mathcal{H}_{\mathrm{loc}}(\mathbb{R}^d)$. Moreover, since once continuously differentiable vector potentials are included, in symbols $\left(\mathcal{C}^1(\mathbb{R}^d)\right)^d \subset \mathcal{H}_{\mathrm{loc}}(\mathbb{R}^d)$, most physically relevant vector potentials are covered. From this point of view the class $\mathcal{H}(\mathbb{R}^d)$ is of less interest, because, for example, any vector potential $A \in \left(\mathcal{C}^1(\mathbb{R}^3)\right)^3$ giving rise to a constant magnetic field $\nabla \times A$ lies in $\mathcal{H}_{\mathrm{loc}}(\mathbb{R}^3)$ but not in $\mathcal{H}(\mathbb{R}^3)$. Therefore, we try to avoid global regularity assumptions as far as possible, that is, we aspire after results valid for vector potentials in $\mathcal{H}_{\mathrm{loc}}(\mathbb{R}^d)$.

ii) With regard to the one-parameter family of vector potentials considered in [13, Theorem 6.2], we note that the spectrum of the associated Schrödinger operator dramatically changes its character precisely at that parameter value where the border between $\mathcal{H}(\mathbb{R}^d)$ and $\mathcal{H}_{\mathrm{loc}}(\mathbb{R}^d)$ is reached.

iii) The local singularities admissible for vector potentials in $\mathcal{H}(\mathbb{R}^d)$, $d \geq 3$, are illustrated by the example

$$A_\mu(x) := \frac{x}{|x|} \left(V_\mu(x)\right)^{1/2}, \tag{2.17}$$

where V_μ is defined in (2.13). According to Remark 2.2.*iv)* one has $A_\mu^2 \in \mathcal{K}(\mathbb{R}^d) \Leftrightarrow \mu > 1$, but

$$A_\mu \in \mathcal{H}(\mathbb{R}^d) \Leftrightarrow \mu > 2, \tag{2.18}$$

as can be seen by explicitly calculating $\nabla \cdot A_\mu$.

Similar to Kato-decomposable scalar potentials, vector potentials in $\mathcal{H}_{\mathrm{loc}}(\mathbb{R}^d)$ can be approximated by smooth functions.

Proposition 2.6

Let $A \in \mathcal{H}_{loc}(\mathbb{R}^d)$. Then there is a sequence $\{A_m\}_{m\in\mathbb{N}} \subset \left(\mathcal{C}_0^\infty(\mathbb{R}^d)\right)^d$ such that

$$\lim_{m\to\infty} \left\| (A - A_m)^2 \chi_K \right\|_{\mathcal{K}(\mathbb{R}^d)} = 0 \tag{2.19}$$

and

$$\lim_{m\to\infty} \left\| (\nabla \cdot A - \nabla \cdot A_m) \chi_K \right\|_{\mathcal{K}(\mathbb{R}^d)} = 0 \tag{2.20}$$

for all compact $K \subset \mathbb{R}^d$.

The proof is similar to the one of Proposition 2.3 and is given in Appendix A.

The next topic is the precise construction of the **Schrödinger operator** (1.1) as a self-adjoint operator acting on the Hilbert space $\mathrm{L}^2(\Lambda)$. Here we use the definition via forms [51, Section VIII.6], [50, Section X.3], [13, Section 1.1].

In a first step we consider $A \in \mathcal{H}_{\mathrm{loc}}(\mathbb{R}^d)$ and $V^+ \in \mathcal{K}_{\mathrm{loc}}(\mathbb{R}^d)$. The sesquilinear form

$$h_\Lambda^{A,V^+} : \mathcal{C}_0^\infty(\Lambda) \times \mathcal{C}_0^\infty(\Lambda) \to \mathbb{C}, \quad (\phi, \psi) \mapsto h_\Lambda^{A,V^+}(\phi, \psi) \tag{2.21}$$

where

$$h_\Lambda^{A,V^+}(\phi, \psi) := \langle (V^+)^{\frac{1}{2}}\phi \,|\, (V^+)^{\frac{1}{2}}\psi \rangle + \frac{1}{2}\sum_{j=1}^d \langle (-i\nabla - A)_j\phi \,|\, (-i\nabla - A)_j\psi \rangle \tag{2.22}$$

is densely defined in $\mathrm{L}^2(\Lambda)$ and non-negative. For $\Lambda = \mathbb{R}^d$ the closure of the form (2.21) has form domain

$$\mathcal{Q}\left(h_{\mathbb{R}^d}^{A,V^+}\right) := \left\{\psi \in \mathrm{L}^2(\mathbb{R}^d) : (-i\nabla - A)\psi \in \left(\mathrm{L}^2(\mathbb{R}^d)\right)^d, \, (V^+)^{\frac{1}{2}}\psi \in \mathrm{L}^2(\mathbb{R}^d)\right\} \tag{2.23}$$

see [57], [13, Theorem 1.13]. For general open $\Lambda \subseteq \mathbb{R}^d$ the completion $\mathcal{Q}\left(h_\Lambda^{A,V^+}\right)$ of $\mathcal{C}_0^\infty(\Lambda)$ with respect to the form norm

$$\|\psi\|_{h_\Lambda^{A,V^+}} := \sqrt{\|\psi\|_2^2 + h_\Lambda^{A,V^+}(\psi, \psi)} \tag{2.24}$$

cannot exceed $\mathrm{L}^2(\mathbb{R}^d)$ because the form is closable for $\Lambda = \mathbb{R}^d$. Since $\mathrm{L}^2(\Lambda)$ is a closed subspace of $\mathrm{L}^2(\mathbb{R}^d)$ the completion is even a subspace of $\mathrm{L}^2(\Lambda)$. Thus the form (2.21) extended to the domain $\mathcal{Q}\left(h_\Lambda^{A,V^+}\right)$ is non-negative and closed. According to [51, Theorem VIII.15] it therefore defines a unique self-adjoint operator on $\mathrm{L}^2(\Lambda)$ denoted as $H_\Lambda(A, V^+)$.

Before proceeding with the construction of the Schrödinger operator we note that the just-defined Schrödinger operator $H_\Lambda(A, 0)$, $A \in \mathcal{H}_{\mathrm{loc}}(\mathbb{R}^d)$, obeys the so-called diamagnetic inequality

$$|e^{-tH_\Lambda(A,0)}\psi| \le e^{-tH_\Lambda(0,0)}|\psi|, \quad \psi \in \mathrm{L}^2(\Lambda), \, t \ge 0. \tag{2.25}$$

For $\Lambda = \mathbb{R}^d$ the inequality may be found in [58, Proposition B.13.1], [5, Theorem 2.3]. For general open $\Lambda \subseteq \mathbb{R}^d$ the estimate (2.25) is given in [47, Corollary 3.6] and is a special case of [42, Corollary 2.3]. Moreover, we will get it in the version (2.25) from Lemma B.5 in Appendix B devoted to a proof of the Feynman-Kac-Itô formula below.

In the second step we consider $A \in \mathcal{H}_{\mathrm{loc}}(\mathbb{R}^d)$ and $V \in \mathcal{K}_\pm(\mathbb{R}^d)$. According to Remark 2.2.iii) $V^- \in \mathcal{K}(\mathbb{R}^d)$ is infinitesimally form-bounded relative to $H_\Lambda(0, 0)$. This and the diamagnetic inequality (2.25) imply that V^- is infinitesimally form-bounded relative to $H_\Lambda(A, 0) \le H_\Lambda(A, V^+)$. In the case $\Lambda = \mathbb{R}^d$ the last statement is proven in [5, Theorem 2.5], [56, Theorem 15.10]. For general open $\Lambda \subseteq \mathbb{R}^d$ it is proven in [47, Proposition 3.7]. All three proofs are virtually identical.

Due to the established form-boundedness the KLMN theorem [50, Theorem X.17] is applicable and ensures that

$$h_\Lambda^{A,V} : \mathcal{Q}\left(h_\Lambda^{A,V^+}\right) \times \mathcal{Q}\left(h_\Lambda^{A,V^+}\right) \to \mathbb{C},$$
$$(\phi, \psi) \mapsto h_\Lambda^{A,V}(\phi, \psi) := h_\Lambda^{A,V^+}(\phi, \psi) - \langle (V^-)^{\frac{1}{2}}\phi \,|\, (V^-)^{\frac{1}{2}}\psi \rangle \tag{2.26}$$

is a closed sesquilinear form bounded from below and with form core $\mathcal{C}_0^\infty(\Lambda)$. The associated semi-bounded self-adjoint operator is denoted as $H_\Lambda(A, V)$.

Remarks 2.7

i) Of course, the above construction of $H_\Lambda(A, V)$ works for $A^2, V^+ \in \mathrm{L}_{\mathrm{loc}}^1(\mathbb{R}^d)$ and V^- form-bounded relative to $H_\Lambda(A, V^+)$ with bound strictly smaller than 1. If we even assume $A^2, \nabla \cdot A, V^+ \in \mathrm{L}_{\mathrm{loc}}^2(\mathbb{R}^d)$ the construction of $H_\Lambda(A, V)$ is a bit easier, since then the form (2.21) comes directly from the non-negative symmetric operator

$$\mathcal{C}_0^\infty(\Lambda) \to \mathrm{L}^2(\Lambda), \quad \psi \mapsto \frac{1}{2}\left(-i\nabla - A\right)^2 \psi + V^+ \psi \tag{2.27}$$

and is therefore closable [50, Theorem X.23].

ii) $\mathcal{C}_0^\infty(\mathbb{R}^d)$ is not only a form core but even an operator core for $H_{\mathbb{R}^d}(A, V)$, if in addition to $A \in \mathcal{H}_{\mathrm{loc}}(\mathbb{R}^d)$, $V \in \mathcal{K}_\pm(\mathbb{R}^d)$ one has $A^2, \nabla \cdot A, V \in \mathrm{L}_{\mathrm{loc}}^2(\mathbb{R}^d)$. This follows from [31, Theorem 2.5]. In [58, Theorem B.13.4] the statement without the restriction $\nabla \cdot A \in \mathrm{L}_{\mathrm{loc}}^2(\mathbb{R}^d)$ is incorrectly ascribed to [40]. Other criteria for the essential self-adjointness of $H_{\mathbb{R}^d}(A, V)$ on $\mathcal{C}_0^\infty(\mathbb{R}^d)$ may be found for instance in [40, Theorem 3] and, more generally, in [41, Corollary 1.4].

iii) If $\Lambda \neq \mathbb{R}^d$ the above construction of $H_\Lambda(A, V)$ corresponds, roughly speaking, to imposing Dirichlet boundary conditions on $\partial\Lambda$ in order to render (1.1) formally self-adjoint. In fact, for $A = 0$ one has $\mathcal{Q}\left(h_\Lambda^{0,V}\right) = \mathcal{Q}(V^+) \cap \overset{\circ}{\mathrm{H}}{}^1(\Lambda)$ where $\overset{\circ}{\mathrm{H}}{}^1(\Lambda)$ is the Sobolev space of functions in $\mathrm{L}^2(\Lambda)$ vanishing on $\partial\Lambda$ in distributional sense and having a square-integrable distributional gradient, see, for example, [20, Section 2.3].

We are going to recall the Feynman-Kac-Itô formula for the semigroup $\{\mathrm{e}^{-tH_\Lambda(A, V)}\}_{t \geq 0}$ generated by the above-defined Schrödinger operator $H_\Lambda(A, V)$.

We denote by $\mathbb{P}_x$ Wiener's probability measure associated with standard **Brownian motion** w on $\mathbb{R}^d$ having diffusion constant $\frac{1}{2}$ and almost surely continuous paths $s \mapsto w(s)$ starting from $x \in \mathbb{R}^d$, that is, $w(0) = x$. The induced expectation is written as $\mathbb{E}_x$.

According to [2, Theorem 4.5] the Kato class is conveniently characterized in terms of Brownian motion by the following equivalence

$$f \in \mathcal{K}(\mathbb{R}^d) \Leftrightarrow \lim_{t \downarrow 0} \sup_{x \in \mathbb{R}^d} \mathbb{E}_x\left[\int_0^t |f(w(s))| \, \mathrm{d}s\right] = 0. \tag{2.28}$$

This is a consequence of the chain of inequalities

$$C_1(d) \int g_{t^{\kappa(d)}}(x - y) |f(y)| \, \mathrm{d}y$$

$$\leq \mathbb{E}_x\left[\int_0^t |f(w(s))| \, \mathrm{d}s\right] \tag{2.29}$$

$$\leq C_2(d) \int g_\varrho(x - y) |f(y)| \, \mathrm{d}y + C_3(d, \varrho, t) \|f\|_{\mathrm{L}_{\mathrm{unif,\ loc}}^1(\mathbb{R}^d)},$$

which hold for all $d \geq 2$, $0 < t < 1$, $0 < \varrho < \frac{1}{2}$. Here we have set $\kappa(2) := 1$ and $\kappa(d) := \frac{1}{2}$ for $d \geq 3$ and $C_1(d), C_2(d)$ are strictly positive constants. Moreover, C_3 is a function obeying $\lim_{t \downarrow 0} C_3(d, \varrho, t) = 0$ for all $d \geq 2$, $0 < \varrho < \frac{1}{2}$. This chain of inequalities is implicit in the proof of [2, Theorem 4.5].

Since, almost surely, Brownian-motion paths stay for finite times in bounded regions, the equivalence (2.28) proves the implication

$$f \in \mathcal{K}_{\mathrm{loc}}(\mathbb{R}^d) \Rightarrow \mathbb{P}_x\left\{\int_0^t |f(w(s))|\,\mathrm{d}s < \infty\right\} = 1, \quad x \in \mathbb{R}^d,\ t \geq 0. \tag{2.30}$$

Confer the proof of Lemma C.8 of Appendix C.

Remarks 2.8

   i) The implication (2.30) explains with the help of [25, Section 4.3] or [36, Definition 3.2.23] that the stochastic line integral in the sense of Itô

$$t \mapsto \int_0^t A(w(s)) \cdot \mathrm{d}w(s) \tag{2.31}$$

of a vector potential $A \in \mathcal{H}_{\mathrm{loc}}(\mathbb{R}^d)$ is for all $x = w(0) \in \mathbb{R}^d$ a well-defined stochastic process possessing a continuous version.

   ii) For a vector potential $A \in \mathcal{H}_{\mathrm{loc}}(\mathbb{R}^d)$ and a scalar potential $V \in \mathcal{K}_{\pm}(\mathbb{R}^d)$ the potentials' part of the Euclidean **action**

$$t \mapsto S_t(A, V|w), \quad t \geq 0, \tag{2.32}$$

where

$$S_t(A, V|w) := i \int_0^t A(w(s)) \cdot \mathrm{d}w(s) + \frac{i}{2} \int_0^t (\nabla \cdot A)(w(s))\,\mathrm{d}s$$
$$+ \int_0^t V(w(s))\,\mathrm{d}s \tag{2.33}$$

is for all $x = w(0) \in \mathbb{R}^d$ a well-defined complex-valued stochastic process possessing a continuous version. This follows from the preceding remark and (2.30), confer [10, Lemma 3.1].

Our basic tool will be the following variant of the **Feynman-Kac-Itô formula**.

Proposition 2.9

Let $A \in \mathcal{H}_{loc}(\mathbb{R}^d)$, $V \in \mathcal{K}_{\pm}(\mathbb{R}^d)$ and $\Lambda \subseteq \mathbb{R}^d$ open. Then for any $\psi \in \mathrm{L}^2(\Lambda)$ and $t \geq 0$ one has

$$\left(\mathrm{e}^{-tH_\Lambda(A,V)}\psi\right)(x) = \mathbb{E}_x\left[\mathrm{e}^{-S_t(A,V|w)}\,\Xi_{\Lambda,t}(w)\,\psi(w(t))\right] \tag{2.34}$$

for almost all $x \in \Lambda$, where

$$\Xi_{\Lambda,t}(w) := \begin{cases} 1 & \text{if } w(s) \in \Lambda \text{ for all } 0 < s \leq t \\ 0 & \text{otherwise.} \end{cases} \tag{2.35}$$

For $\Lambda = \mathbb{R}^d$, $\nabla \cdot A = 0$ the proposition is a special case of [56, Theorem 15.5]. The proof given there can be extended easily to $\Lambda = \mathbb{R}^d$, $\nabla \cdot A \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$. For $\Lambda \subseteq \mathbb{R}^d$, $A = 0$, $V = 0$ the proposition is equivalent to [54, Theorem 3] and [56, Theorem 21.1]. Finally, for $\Lambda \subseteq \mathbb{R}^d$, $A = 0$, $V^+ \in \mathcal{K}_{\mathrm{loc}}(\mathbb{R}^d)$, $V^- = 0$, the proposition follows from [64, Proposition 1.3.3, Theorem 1.4.11]. We have not found a proof for the above setting in the literature. Therefore we give a detailed proof in Appendix B. There we use Proposition 2.9 for $\Lambda = \mathbb{R}^d$ to get (2.34) for $\Lambda \subset \mathbb{R}^d$, $A \in \mathcal{H}_{\mathrm{loc}}(\mathbb{R}^d)$ and $V \in L^\infty(\mathbb{R}^d)$ in a way patterned after the proof of [54, Theorem 3], [56, Theorem 21.1]. Eventually we generalize the achieved result to the assumptions of the proposition by an approximation technique taken from the proof of [56, Theorem 6.2].

With considerably more effort it is possible to prove a Feynman-Kac-Itô formula for still more general vector potentials, see [34, 35]. There is even a Feynman-Kac-Itô formula under virtually minimal conditions [47], which suffers, however, from being less explicit than (2.34).

From the triangle inequality and the proof of [10, Proposition 3.1] or [58, Theorem B.1.1] the right-hand side of (2.34) – considered as a function of x – is seen to lie in $L^q(\Lambda)$ for any $\psi \in L^p(\Lambda)$ where $t > 0$, $1 \leq p \leq q \leq \infty$. This statement is also a special case of Lemma C.1. Hence the right-hand side of (2.34) can be used to define an operator $T^t_{p,q} : L^p(\Lambda) \to L^q(\Lambda)$. By the same argument this operator is bounded, that is,

$$\left\| T^t_{p,q} \right\|_{p,q} < \infty, \quad t > 0, \; 1 \leq p \leq q \leq \infty. \tag{2.36}$$

The family of operators $\{T^t_{p,q}\}_{t \geq 0, \, 1 \leq p \leq q \leq \infty}$ thus obtained constitutes a one-parameter **semigroup** in the sense that

$$T^0_{p,p} = 1, \quad T^{s+t}_{p,r} = T^s_{q,r} \circ T^t_{p,q}, \quad s, t > 0, \; 1 \leq p \leq q \leq r \leq \infty. \tag{2.37}$$

This follows from the Markov properties of Brownian motion and of the Itô integral in (2.33). Moreover, the family is **self-adjoint** in the sense that the dual mapping of $T^t_{p,q}$ is $T^t_{p',q'}$ where the indices p', q' dual to p, q are defined as usual through $\frac{1}{p'} + \frac{1}{p} := 1 =: \frac{1}{q'} + \frac{1}{q}$.

In the next section we will see that $\{T^t_{p,p}\}_{t \geq 0}$ is a strongly continuous semigroup for all finite $1 \leq p < \infty$. Nevertheless, for mnemonic reasons, we will always write $\mathrm{e}^{-tH_\Lambda(A,V)}$ instead of $T^t_{p,q}$ for all $1 \leq p \leq q \leq \infty$.

Within this notation the triangle inequality applied to the right-hand side of (2.34) generalizes the **diamagnetic inequality** (2.25) to

$$\left| \mathrm{e}^{-tH_\Lambda(A,V)} \psi \right| \leq \mathrm{e}^{-tH_\Lambda(0,V)} |\psi|, \quad \psi \in L^p(\Lambda), \tag{2.38}$$

confer [58, Theorem B.13.2], [42, Equation (2.13)].

For vanishing magnetic field $A = 0$ the semigroup is increasing in Λ in the sense that for $\Lambda \subseteq \Lambda' \subseteq \mathbb{R}^d$, Λ, Λ' open, one has

$$\mathrm{e}^{-tH_\Lambda(0,V)} \chi_\Lambda \psi \leq \mathrm{e}^{-tH_{\Lambda'}(0,V)} \psi \tag{2.39}$$

for all $\psi \geq 0$, $\psi \in L^p(\Lambda')$, $1 \leq p \leq \infty$ and $t \geq 0$. Because $\Xi_{\Lambda,t} \leq \Xi_{\Lambda',t}$, this follows from (2.34) by inspection.

The above-mentioned boundedness of $\mathrm{e}^{-tH_\Lambda(A,V)}$ can now be sharpened to the statement that for given $V \in \mathcal{K}_\pm(\mathbb{R}^d)$ there are real constants C and E, independent of t, p

and q, so that

$$
\begin{aligned}
\left\|\mathrm{e}^{-tH_\Lambda(A,V)}\right\|_{p,q} &\leq \left\|\mathrm{e}^{-tH_\Lambda(0,V)}\right\|_{p,q} \leq \left\|\mathrm{e}^{-tH_{\mathbb{R}^d}(0,V)}\right\|_{p,q} \\
&\leq \begin{cases} C\,t^{-(q-p)d/2pq}\,\mathrm{e}^{-tE} & 1 \leq p < q \leq \infty \\ C\,\mathrm{e}^{-tE} & 1 \leq p = q \leq \infty \end{cases} \quad \text{for}
\end{aligned}
\tag{2.40}
$$

for all $A \in \mathcal{H}_{\mathrm{loc}}(\mathbb{R}^d)$. The first inequality follows from (2.38), confer [58, Corollary B.13.3], the middle inequality is a consequence of (2.39) and the last inequality is given in [58, Equation (B.11)]. For related estimates see also [2, Section 1] and references therein. Furthermore, C and E can be chosen [58, Theorem B.5.1] such that E is any number smaller than the infimum of the $\mathrm{L}^2(\mathbb{R}^d)$-spectrum of $H_{\mathbb{R}^d}(0,V)$.

Eventually, we turn to the notion of a configuration space with a regular boundary.

Definition 2.10

*A set $\Lambda \subset \mathbb{R}^d$ is called **regular**, if it is open and*

$$
\mathbb{E}_x[\Xi_{\Lambda,t}(w)] = 0
\tag{2.41}
$$

*for all $x \in \partial\Lambda$, $t > 0$. The set $\Lambda \subset \mathbb{R}^d$ is called **uniformly regular**, if it is open and*

$$
\lim_{\tau \downarrow 0} \sup_{x \in \partial\Lambda} \mathbb{E}_x[\Xi_{\Lambda,t-\tau}(w(\bullet + \tau))] = 0
\tag{2.42}
$$

for all $t > 0$. Furthermore, it is understood that $\Lambda = \mathbb{R}^d$ is both regular and uniformly regular.

Remarks 2.11

i) The above definition of regularity is equivalent to the standard one [6, Definition II.1.9], [48, Section 2.3], which employs first exit times. More precisely, $\Lambda \subset \mathbb{R}^d$ is regular if and only if it is open and

$$
\mathbb{P}_x\{\inf\{s > 0 : w(s) \notin \Lambda\} = 0\} = 1
\tag{2.43}
$$

for all $x \in \partial\Lambda$. To prove this assertion we first mention that by the definition (2.35) of $\Xi_{\Lambda,t}(w)$, (2.43) implies (2.41). To show the opposite direction, note that

$$
t \mapsto \{w(s) \in \Lambda \text{ for all } 0 < s \leq t\}
\tag{2.44}
$$

is decreasing in the sense of set inclusion. Therefore, (2.43) is equivalent to

$$
\mathbb{P}_x\left\{ \bigcup_{t>0, t\in\mathbb{Q}} \{w(s) \in \Lambda \text{ for all } 0 < s \leq t\} \right\} = 0,
\tag{2.45}
$$

where $\mathbb{Q}$ denotes the set of rational numbers. Due to (2.35), (2.41) implies (2.45).

ii) Using dominated convergence, one checks that (2.41) is equivalent to

$$\lim_{\tau \downarrow 0} \mathbb{E}_x[\Xi_{\Lambda, t-\tau}(w(\bullet + \tau))] = 0 \qquad (2.46)$$

for all $x \in \partial\Lambda$, $t > 0$. Therefore, regularity of Λ is implied by uniform regularity, as it should.

There are several known conditions implying regularity, see, for example, [6, Section II.1], [48, Section 2.3]. Here we only recall **Poincaré's cone condition**, because it may easily be adapted to uniform regularity. The open set

$$C_{r,\beta}(x, u) := \left\{ y \in \mathbb{R}^d : 0 < |y - x| < r, \, 0 < u \cdot (y - x) < |y - x| \cos\beta \right\} \qquad (2.47)$$

is called a **finite cone** with vertex at $x \in \mathbb{R}^d$ in direction $u \in \mathbb{R}^d$, $|u| = 1$, with opening angle $0 < \beta < \frac{\pi}{2}$ and radius $r > 0$.

Proposition 2.12

Let $\Lambda \subset \mathbb{R}^d$ open.

i) *If for all $x \in \partial\Lambda$ there is a finite cone $C_{r,\beta}(x, u) \subset \mathbb{R}^d \backslash \Lambda$, $u \in \mathbb{R}^d$, $|u| = 1$, $r > 0$, $0 < \beta < \frac{\pi}{2}$, then Λ is regular.*

ii) *If there are constants $0 < \beta < \frac{\pi}{2}$, $r > 0$ such that for all $x \in \partial\Lambda$ there is a finite cone $C_{r,\beta}(x, u) \subset \mathbb{R}^d \backslash \Lambda$, $u \in \mathbb{R}^d$, $|u| = 1$, then Λ is uniformly regular.*

Proof:

The first assertion is proven in [6] as Proposition 1.13. This may be seen, using Remark 2.11.*i)*.

To show the second part, we use for $x \in \partial\Lambda$ the estimate

$$\begin{aligned} \mathbb{E}_x[\Xi_{\Lambda, t-\tau}(w(\bullet + \tau))] &\leq \mathbb{E}_x\left[\Xi_{\mathbb{R}^d \backslash \overline{C_{r,\beta}(x,u)}, t-\tau}(w(\bullet + \tau)) \right] \\ &= \mathbb{E}_0\left[\Xi_{\mathbb{R}^d \backslash \overline{C_{r,\beta}(0,\overline{u})}, t-\tau}(w(\bullet + \tau)) \right], \end{aligned} \qquad (2.48)$$

where the equality follows from the rotation and translation invariance of Brownian motion and $\overline{u}$ is any given unit vector. By the first part of the proposition, $\mathbb{R}^d \backslash \overline{C_{r,\beta}(0,\overline{u})}$ is regular. Thus the right-hand side of (2.48), which is independent of $x \in \mathbb{R}^d$, tends to 0 as $\tau \downarrow 0$ due to Remark 2.11.*ii)*. This proves (2.42), whence uniform regularity. ∎

3 Continuity of the semigroup in its parameter

The following result is a straightforward generalization of [10, Proposition 3.2] to non-zero magnetic fields and $\Lambda \subseteq \mathbb{R}^d$.

Theorem 3.1

Let $A \in \mathcal{H}_{loc}(\mathbb{R}^d)$, $V \in \mathcal{K}_{\pm}(\mathbb{R}^d)$ and $\Lambda \subseteq \mathbb{R}^d$ open. Moreover, let $1 \leq p < \infty$ be finite. Then the semigroup

$$\left\{ \mathrm{e}^{-tH_\Lambda(A,V)} : \ \mathrm{L}^p(\Lambda) \to \mathrm{L}^p(\Lambda) \right\}_{t \geq 0} \tag{3.1}$$

as defined in Section 2 is **strongly continuous**, that is,

$$\lim_{t' \to t} \left\| \left(\mathrm{e}^{-tH_\Lambda(A,V)} - \mathrm{e}^{-t'H_\Lambda(A,V)} \right) \psi \right\|_p = 0 \tag{3.2}$$

for all $\psi \in \mathrm{L}^p(\Lambda)$ and all $t \geq 0$.

Proof:

Due to the semigroup property (2.37) for $p = q = r$ we may assume $0 \leq t,\, t' \leq 1$. Then the norm in (3.2) is bounded from above by

$$\left(\sup_{0 \leq t \leq 1} \left\| \mathrm{e}^{-tH_\Lambda(A,V)} \right\|_{p,p} \right) \left\| \left(\mathrm{e}^{-|t-t'|H_\Lambda(A,V)} - 1 \right) \psi \right\|_p . \tag{3.3}$$

Using (2.40) it is therefore sufficient to show (3.2) for $t = 0$. Since (3.2) holds in the free case $A = 0$, $V = 0$ and $\Lambda = \mathbb{R}^d$, it is enough to establish

$$\lim_{t \downarrow 0} \left\| D_{t,t} \psi \right\|_p = 0. \tag{3.4}$$

Here we have made use of the abbreviation

$$D_{t,\tau} := \mathrm{e}^{-\tau H_{\mathbb{R}^d}(0,0)} \mathrm{e}^{-(t-\tau)H_\Lambda(A,V)} - \mathrm{e}^{-tH_\Lambda(A,V)} \tag{3.5}$$

where $0 \leq \tau \leq t$. The Feynman-Kac-Itô formula (2.34) and the Jensen inequality $|\mathbb{E}_x[\bullet]|^p \leq \mathbb{E}_x[|\bullet|^p]$ give

$$|(D_{t,t}\psi)(x)|^p \leq \mathbb{E}_x\left[\left| 1 - \mathrm{e}^{-S_t(A,V|w)} \Xi_{\Lambda,t}(w) \right|^p |\psi(w(t))|^p \right]. \tag{3.6}$$

Exploiting that $S_t(A,V|w)$ turns into its complex conjugate under time reversal of Brownian motion, (3.6) leads upon integration over $x \in \Lambda$ to

$$\left(\left\| D_{t,t}\psi \right\|_p \right)^p \leq \int_\Lambda \mathbb{E}_x\left[\left| 1 - \mathrm{e}^{-S_t(A,V|w)} \Xi_{\Lambda,t}(w) \right|^p \right] |\psi(x)|^p \ \mathrm{d}x. \tag{3.7}$$

Using $|z - z'|^p \leq 2^p(|z|^p + |z'|^p)$ for $z, z' \in \mathbb{C}$, we obtain

$$\begin{aligned}
\left(\left\| D_{t,t}\psi \right\|_p \right)^p \leq \ &2^p \int_\Lambda \mathbb{E}_x\left[\left| 1 - \mathrm{e}^{-S_t(A,V|w)} \right|^p \right] |\psi(x)|^p \ \mathrm{d}x. \\
&+ 2^p \int_\Lambda \mathbb{E}_x\left[1 - \Xi_{\Lambda,t}(w) \right] |\psi(x)|^p \ \mathrm{d}x
\end{aligned} \tag{3.8}$$

and employing additionally $-V \leq V^-$ and $t \leq 1$ we get

$$\left| 1 - \mathrm{e}^{-S_t(A,V|w)} \right|^p \leq 2^p \left(1 + \mathrm{e}^{-S_1(0,-pV^-|w)} \right). \tag{3.9}$$

The action $S_t(A, V | w)$ vanishes for all $x = w(0)$ almost surely as $t \downarrow 0$ due to Remark 2.8.*ii)*. Moreover,

$$\int_\Lambda \mathbb{E}_x \left[1 + e^{-S_1(0, -pV^- | w)} \right] |\psi(x)|^p \, dx \leq \left(1 + \left\| e^{-H_{\mathbb{R}^d}(0, -pV^-)} \right\|_{\infty,\infty} \right) (\|\psi\|_p)^p < \infty \quad (3.10)$$

by (2.34) and (2.40). Hence the dominated-convergence theorem implies that the first integral on the right-hand side of (3.8) vanishes as $t \downarrow 0$. In order to show that the second integral vanishes too, we claim

$$\lim_{t \downarrow 0} \sup_{x \in \Lambda_r} \mathbb{E}_x[1 - \Xi_{\Lambda,t}(w)] = 0 \quad (3.11)$$

for all $r > 0$. Here

$$\Lambda_r := \{x \in \Lambda : |x - y| > r \text{ for all } y \in \partial\Lambda\} \quad (3.12)$$

denotes the set of points well inside Λ. In fact, one has

$$\sup_{x \in \Lambda_r} \mathbb{E}_x[1 - \Xi_{\Lambda,t}(w)] \leq \mathbb{P}_0 \left\{ \sup_{0 < s \leq t} |w(s)| \geq r \right\} \quad (3.13)$$

and the right-hand side vanishes as $t \downarrow 0$ due to Lévy's maximal inequality [56, Equation (7.6')]. This completes the proof of (3.4). ∎

Remarks 3.2

i) Even for $A = 0$ and $\Lambda = \mathbb{R}^d$ Theorem 3.1 is slightly different from [10, Proposition 3.2] as can be inferred from (2.11). For $A = 0$ and $\Lambda \subseteq \mathbb{R}^d$ with finite Lebesgue measure Theorem 3.1 is contained in [11, Theorem 3.17].

ii) As we will see in the next section, the set $e^{-tH_\Lambda(A,V)}L^\infty(\Lambda)$ contains only continuous functions if $A \in \mathcal{H}_{loc}(\mathbb{R}^d)$, $V \in \mathcal{K}_\pm(\mathbb{R}^d)$ and $t > 0$. Therefore, the semigroup $\left\{ e^{-tH_\Lambda(A,V)} : L^\infty(\Lambda) \to L^\infty(\Lambda) \right\}_{t \geq 0}$ is not strongly continuous for all pairs $(A, V) \in \mathcal{H}_{loc}(\mathbb{R}^d) \times \mathcal{K}_\pm(\mathbb{R}^d)$, see also [10, Remark 3.4]. However, for $\Lambda = \mathbb{R}^d$, consider the closed subspace $\mathcal{C}_\infty(\mathbb{R}^d)$ of $L^\infty(\mathbb{R}^d)$ consisting of continuous functions vanishing at infinity. A slight modification of the proof of (4.6) below shows that the semigroup maps $\mathcal{C}_\infty(\mathbb{R}^d)$ into itself. Moreover, with somewhat more effort it can be shown that this restriction yields a strongly continuous semigroup, confer [11, Theorem 3.17].

4 Continuity of the image functions of the semigroup

In this section we prove that the operator $e^{-tH_\Lambda(A,V)}$ is smoothing in the sense that it maps $L^p(\Lambda)$ into the set $\mathcal{C}(\Lambda)$ of complex-valued continuous functions on Λ.

For the reader's convenience we recall that a family $\mathcal{F}$ of functions $f : \Lambda \to \mathbb{C}$ is called **equicontinuous**, if

$$\lim_{x' \to x} \sup_{f \in \mathcal{F}} |f(x) - f(x')| = 0 \quad (4.1)$$

for all $x \in \Lambda$ and it is called **uniformly equicontinuous**, if

$$\lim_{r \downarrow 0} \sup_{x, x' \in \Lambda, |x - x'| < r} \sup_{f \in \mathcal{F}} |f(x) - f(x')| = 0, \tag{4.2}$$

see, for example, [51, Section I.6].

Theorem 4.1

Let $A \in \mathcal{H}_{loc}(\mathbb{R}^d)$, $V \in \mathcal{K}_{\pm}(\mathbb{R}^d)$ and $\Lambda \subseteq \mathbb{R}^d$ open. Then

$$\mathrm{e}^{-tH_\Lambda(A,V)} \, \mathrm{L}^p(\Lambda) \subseteq \mathrm{L}^q(\Lambda) \cap \mathcal{C}(\Lambda), \quad t > 0, \ 1 \le p \le q \le \infty, \tag{4.3}$$

and for fixed $t > 0$, $1 \le p \le \infty$ the family

$$\left\{ \mathrm{e}^{-tH_\Lambda(A,V)} \, \psi : \psi \in \mathrm{L}^p(\Lambda), \ \|\psi\|_p \le 1 \right\} \tag{4.4}$$

of functions on Λ is equicontinuous. Furthermore, the right-hand side of (2.34) gives the continuous representative of

$$x \mapsto \left(\mathrm{e}^{-tH_\Lambda(A,V)} \psi \right)(x), \quad x \in \Lambda. \tag{4.5}$$

Finally,

$$\lim_{|x| \to \infty} \left(\mathrm{e}^{-tH_\Lambda(A,V)} \psi \right)(x) = 0 \tag{4.6}$$

for all $\psi \in \mathrm{L}^p(\Lambda)$ with finite $1 \le p < \infty$ and all $t > 0$.

Proof:

Using the boundedness (2.40) and the semigroup property (2.37) with $q = r = \infty$ it is sufficient to prove that (4.4) is an equicontinuous family for $p = \infty$ in order to get both (4.3) and the equicontinuity for all $1 \le p \le \infty$. Since this holds in the free case $A = 0$, $V = 0$ and $\Lambda = \mathbb{R}^d$, one has due to (2.40) that

$$\left\{ \mathrm{e}^{-\tau H_{\mathbb{R}^d}(0,0)} \, \mathrm{e}^{-(t-\tau)H_\Lambda(A,V)} \psi : \psi \in \mathrm{L}^\infty(\Lambda), \ \|\psi\|_\infty \le 1 \right\} \tag{4.7}$$

is an equicontinuous family for all $0 < \tau \le t$. Therefore, it is enough to show

$$\lim_{\tau \downarrow 0} \operatorname{ess\,sup}_{x \in K} \sup_{\|\psi\|_\infty \le 1} |(D_{t,\tau}\psi)(x)| = 0 \tag{4.8}$$

for all compact $K \subset \Lambda$. To this end, we represent the image of ψ by the operator difference (3.5) as

$$\begin{aligned}
(D_{t,\tau}\psi)(x) = \mathbb{E}_x \Big[& \mathrm{e}^{S_\tau(A,V|w) - S_t(A,V|w)} \\
& \times \left(\Xi_{\Lambda,t-\tau}(w(\tau + \bullet)) - \mathrm{e}^{-S_\tau(A,V|w)} \Xi_{\Lambda,t}(w) \right) \psi(w(t)) \Big]
\end{aligned} \tag{4.9}$$

where we have used the Feynman-Kac-Itô formula (2.34), the additivity of the integrals in the action (2.33) and the Markov property of the Brownian motion w. We use the right-hand side of (4.9) to give meaning to $(D_{t,\tau}\psi)(x)$ for all $x \in \mathbb{R}^d$. Then, in order to

show that the right-hand side of (2.34) defines the continuous representative of (4.5), it is sufficient to establish

$$\lim_{\tau \downarrow 0} \sup_{x \in K} \sup_{\|\psi\|_\infty \leq 1} |(D_{t,\tau}\psi)(x)| = 0 \tag{4.10}$$

for all compact $K \subset \Lambda$.

The triangle inequality in combination with

$$S_\tau(0, V|w) - S_t(0, V|w) \leq -S_t(0, -V^-|w) \tag{4.11}$$

yields

$$|(D_{t,\tau}\psi)(x)| \leq \mathbb{E}_x\Big[\, \mathrm{e}^{-S_t(0,-V^-|w)} \\ \times \big|\Xi_{\Lambda,t-\tau}(w(\tau + \bullet)) - \mathrm{e}^{-S_\tau(A,V|w)}\, \Xi_{\Lambda,t}(w)\big|\; |\psi(w(t))|\,\Big]. \tag{4.12}$$

By $|\psi(w(t))| \leq \|\psi\|_\infty$ and the Cauchy-Schwarz inequality one arrives at

$$|(D_{t,\tau}\psi)(x)|^2 \leq (\|\psi\|_\infty)^2\, \mathbb{E}_x\Big[\mathrm{e}^{-S_t(0,-2V^-|w)}\Big] \\ \times \bigg(\; \mathbb{E}_x\big[\,\Xi_{\Lambda,t-\tau}(w(\tau + \bullet)) - \Xi_{\Lambda,t}(w)\,\big] \\ + \mathbb{E}_x\Big[\big|1 - \mathrm{e}^{-S_\tau(A,V|w)}\big|^2\Big]\;\bigg). \tag{4.13}$$

Since

$$\mathbb{E}_x\big[\,\Xi_{\Lambda,t-\tau}(w(\tau + \bullet)) - \Xi_{\Lambda,t}(w)\,\big] = \mathbb{E}_x\big[\,\Xi_{\Lambda,t-\tau}(w(\tau + \bullet))\,(1 - \Xi_{\Lambda,\tau}(w))\,\big] \\ \leq \mathbb{E}_x\big[\,1 - \Xi_{\Lambda,\tau}(w)\,\big], \tag{4.14}$$

(4.10) follows from (3.11) and Lemmas C.2, C.5. This completes the proof of (4.3) and the equicontinuity and identifies the continuous representative.

Finally, for the proof of (4.6) we may assume without loss of generality $1 < p < \infty$ due to (2.40) and the semigroup property. Then (4.6) follows from Hölder's inequality

$$\big|\big(\mathrm{e}^{-tH_\Lambda(A,V)}\psi\big)(x)\big| \leq \left(\big\|\mathrm{e}^{-tH_{\mathbb{R}^d}(0,p'V)}\big\|_{\infty,\infty}\right)^{1/p'} \left(\big(\mathrm{e}^{-tH_{\mathbb{R}^d}(0,0)}\,|\psi|^p\big)(x)\right)^{1/p} \tag{4.15}$$

where $p' := p/(p-1) < \infty$. ∎

Remarks 4.2

    i) The assertion (4.3) reduces for $A = 0$, $V \in \mathcal{K}_\pm(\mathbb{R}^d)$ and $\Lambda = \mathbb{R}^d$ to [58, Corollary B.3.2]. A related result, also for $A = 0$ and $\Lambda = \mathbb{R}^d$, is [10, Propositions 3.1, 3.3]. Our proof is patterned after that of [10, Lemma 3.2] or [11, Propositions 3.11, 3.12] and, in contrast to the strategy in [58], does not use Propositions 2.3 and 2.6.

ii) For $V = 0$, $\nabla \cdot A = 0$ and $A^2 \in \mathcal{K}_{\mathrm{loc}}(\mathbb{R}^d)$ [59, Theorem 3.1] asserts that

$$\mathrm{e}^{-tH_{\mathbb{R}^d}(A,0)}\left(\mathrm{L}^\infty(\mathbb{R}^d) \cap \mathrm{L}^2(\mathbb{R}^d)\right) \subseteq \mathrm{L}^\infty(\mathbb{R}^d) \cap \mathcal{C}(\mathbb{R}^d), \quad t > 0. \tag{4.16}$$

The proof given there, however, applies for all $A^2 \in \mathcal{K}(\mathbb{R}^d)$ but not for all $A^2 \in \mathcal{K}_{\mathrm{loc}}(\mathbb{R}^d)$, because it can happen that

$$\sup_{x \in K} \mathbb{E}_x\left[\int_0^t (A(w(s)))^2 \,\mathrm{d}s\right] = \infty \tag{4.17}$$

although $A^2 \in \mathcal{K}_{\mathrm{loc}}(\mathbb{R}^d)$ and $K \subset \mathbb{R}^d$ compact. Consider the example

$$A(x) = (x_2, -x_1)\ \mathrm{e}^{|x|^4} \tag{4.18}$$

of a vector potential on $\mathbb{R}^2$.

Since $H_\Lambda(A, V)$ is equipped with Dirichlet boundary conditions on $\partial\Lambda$ one would expect that

$$\lim_{x \to y}\left(\mathrm{e}^{-tH_\Lambda(A,V)}\psi\right)(x) = 0, \quad y \in \partial\Lambda, \tag{4.19}$$

for a sufficiently nice boundary of Λ.

Theorem 4.3

Let $A \in \mathcal{H}_{loc}(\mathbb{R}^d)$, $V \in \mathcal{K}_\pm(\mathbb{R}^d)$ and $\Lambda \subset \mathbb{R}^d$ regular. Then (4.19) holds for all $\psi \in \mathrm{L}^p(\Lambda)$, $1 \le p \le \infty$ and $t > 0$. Furthermore, (4.4) is an equicontinuous family of functions on $\overline{\Lambda}$ for all $1 \le p \le \infty$, $t > 0$.

Proof:

The semigroup property (2.37) and (2.40) ensure that it is sufficient to check the case $p = \infty$. The triangle inequality, the Cauchy-Schwarz inequality and (2.34) imply

$$\left|\left(\mathrm{e}^{-tH_\Lambda(A,V)}\psi\right)(x)\right| \le \|\psi\|_\infty \left(\left\|\mathrm{e}^{-tH_{\mathbb{R}^d}(0,2V)}\right\|_{\infty,\infty}\right)^{1/2} \left(\mathbb{E}_x[\Xi_{\Lambda,t}(w)]\right)^{1/2} \tag{4.20}$$

for almost all $x \in \mathbb{R}^d$. Now the assertions follow from Theorem 4.1, (2.40) and Lemma C.7. ∎

Not surprisingly, uniform continuity of the image functions can be achieved for sufficiently regular $\Lambda \subseteq \mathbb{R}^d$ by imposing global regularity conditions for the potentials, thereby, however, possibly excluding physically relevant cases, confer Remark 2.5.*i)*.

Theorem 4.4

Let $A \in \mathcal{H}(\mathbb{R}^d)$, $V \in \mathcal{K}(\mathbb{R}^d)$ and $\Lambda \subseteq \mathbb{R}^d$ uniformly regular. Then (4.4) is a uniformly equicontinuous family of functions on $\overline{\Lambda}$ for all $1 \le p \le \infty$ and $t > 0$.

Proof:

Analogously to the reasoning at the beginning of the proof of Theorem 4.1 it suffices to check

$$\lim_{\tau \downarrow 0} \operatorname{ess\,sup}_{x \in \Lambda_r} \sup_{\|\psi\|_\infty \leq 1} |(D_{t,\tau}\psi)(x)| = 0 \tag{4.21}$$

in order to get uniform equicontinuity on Λ_r, $r > 0$. This follows with the help of (4.13) and (4.14) from (3.11), Lemma C.2 and Lemma C.3. Using Lemma C.7 and the estimate (4.20) we conclude

$$\lim_{r \downarrow 0} \operatorname{ess\,sup}_{x \in \Lambda \setminus \Lambda_r} \sup_{\|\psi\|_\infty \leq 1} \left|\left(e^{-tH_\Lambda(A,V)}\psi\right)(x)\right| = 0, \tag{4.22}$$

which is sufficient to extend the domain of uniform equicontinuity from Λ_r for $r > 0$ to $\overline{\Lambda}$. ∎

Remark 4.5

Proposition 3.1 in [60] is a special case of Theorem 4.4 and (4.6).

5 Continuity of the semigroup in the potentials

As a motivation for this section consider the simple example

$$A_h(x) := (0, x_1 h), \quad h > 0, \tag{5.1}$$

of a vector potential on $\mathbb{R}^2$. Clearly, A_h belongs to $\mathcal{H}_{\mathrm{loc}}(\mathbb{R}^2)$ and gives rise to a constant magnetic field of strength h. The related semigroup

$$\left\{e^{-tH_{\mathbb{R}^2}(A_h,0)} : \mathrm{L}^2(\mathbb{R}^2) \to \mathrm{L}^2(\mathbb{R}^2)\right\}_{t \geq 0}, \tag{5.2}$$

considered as a function of h, is not norm-continuous because

$$\limsup_{h \downarrow 0} \left\| e^{-tH_{\mathbb{R}^2}(A_h,0)} - e^{-tH_{\mathbb{R}^2}(0,0)} \right\|_{2,2} > 0 \tag{5.3}$$

for all $t > 0$. This can be deduced, for example, from [5, Theorem 6.3]. It reflects the fact that the character of the energy spectrum of an electrically charged point-mass in the Euclidean plane changes from purely continuous to pure point, when an arbitrarily low constant magnetic field, perpendicular to the plane, is turned on.

The following theorem, however, shows that under suitable technical assumptions the weaker notion of **local-norm-continuity** holds.

Theorem 5.1

Let $A \in \mathcal{H}_{loc}(\mathbb{R}^d)$, $\{A_m\}_{m \in \mathbb{N}} \subset \mathcal{H}_{loc}(\mathbb{R}^d)$, $V \in \mathcal{K}_\pm(\mathbb{R}^d)$ and $\{V_n\}_{n \in \mathbb{N}} \subset \mathcal{K}_\pm(\mathbb{R}^d)$ such that for all compact $K \subset \mathbb{R}^d$

$$\lim_{m \to \infty} \left\| (A - A_m)^2 \chi_K \right\|_{\mathcal{K}(\mathbb{R}^d)} = 0, \tag{5.4}$$

$$\lim_{m \to \infty} \left\| (\nabla \cdot A - \nabla \cdot A_m) \chi_K \right\|_{\mathcal{K}(\mathbb{R}^d)} = 0 \tag{5.5}$$

and

$$\lim_{n\to\infty} \left\| (V - V_n)\, \chi_K \right\|_{\mathcal{K}(\mathbb{R}^d)} = 0, \tag{5.6}$$

$$\limsup_{\varrho\downarrow 0}\, \sup_{n\in\mathbb{N}}\, \sup_{x\in\mathbb{R}^d} \int g_\varrho(x - y)\, V_n^-(y)\, \mathrm{d}y = 0. \tag{5.7}$$

Moreover, let $\Lambda \subseteq \mathbb{R}^d$ open. Then

$$\lim_{m,n\to\infty}\, \sup_{\tau_1\le t\le\tau_2} \left\| \chi_K \left(\mathrm{e}^{-tH_\Lambda(A,V)} - \mathrm{e}^{-tH_\Lambda(A_m,V_n)} \right) \right\|_{p,q} = 0 \tag{5.8}$$

and

$$\lim_{m,n\to\infty}\, \sup_{\tau_1\le t\le\tau_2} \left\| \left(\mathrm{e}^{-tH_\Lambda(A,V)} - \mathrm{e}^{-tH_\Lambda(A_m,V_n)} \right) \chi_K \right\|_{p,q} = 0 \tag{5.9}$$

for all compact $K \subset \mathbb{R}^d$, $0 < \tau_1 \le \tau_2 < \infty$ and $1 \le p \le q \le \infty$. Furthermore, for $p = q$ one may allow $\tau_1 = 0$.

Proof:

According to the Riesz-Thorin interpolation theorem [50, Theorem IX.17] it is enough to prove the theorem for the three cases $p = q = 1$, $p = q = \infty$ and $p = 1$, $q = \infty$. Moreover, due to the self-adjointness of the semigroup, the assertions (5.8) and (5.9) are equivalent under the combined substitutions $p \mapsto \left(1 - \frac{1}{p}\right)^{-1}$, $q \mapsto \left(1 - \frac{1}{q}\right)^{-1}$. In consequence, it remains to show the following three partial assertions:

1) (5.8) for $p = q = \infty$, $\tau_1 = 0$

2) (5.9) for $p = q = \infty$, $\tau_1 = 0$

3) (5.8) for $p = 1$, $q = \infty$, $\tau_1 > 0$.

We note that (5.8) and (5.9) in the case $p = q$, $\tau_1 = 0$ follow already from the assertions 1) and 2).

As to assertion 1)

Since the Feynman-Kac-Itô formula (2.34) and the triangle inequality give

$$\begin{aligned}
&\left\| \chi_K \left(\mathrm{e}^{-tH_\Lambda(A,V)} - \mathrm{e}^{-tH_\Lambda(A_m,V_n)} \right) \right\|_{\infty,\infty} \\
&\le \sup_{x\in K} \mathbb{E}_x \left[\left| \mathrm{e}^{-S_t(A,V|w)} - \mathrm{e}^{-S_t(A_m,V_n|w)} \right| \right],
\end{aligned} \tag{5.10}$$

the assertion follows from Lemma C.6.

As to assertion 2)

Let B_R be the open ball of radius $R > 0$ centered about the origin in $\mathbb{R}^d$, see (2.1). Then the diamagnetic inequality (2.38) in combination with (2.39) and the triangle inequality yields

$$
\begin{aligned}
&\left\| \left(\mathrm{e}^{-tH_\Lambda(A,V)} - \mathrm{e}^{-tH_\Lambda(A_m,V_n)} \right) \chi_K \right\|_{\infty,\infty} \\
&\leq \left\| \chi_{B_R} \left(\mathrm{e}^{-tH_\Lambda(A,V)} - \mathrm{e}^{-tH_\Lambda(A_m,V_n)} \right) \right\|_{\infty,\infty} \\
&\quad + \left\| \left(1 - \chi_{B_R} \right) \mathrm{e}^{-tH_{\mathbb{R}^d}(0,V)} \chi_K \right\|_{\infty,\infty} + \left\| \left(1 - \chi_{B_R} \right) \mathrm{e}^{-tH_{\mathbb{R}^d}(0,V_n)} \chi_K \right\|_{\infty,\infty}.
\end{aligned}
\tag{5.11}
$$

Hence assertion 2 follows from assertion 1 provided that

$$
\lim_{R\to\infty} \sup_{0\leq t\leq \tau_2} \sup_{n\in\mathbb{N}} \left\| \left(1 - \chi_{B_R} \right) \mathrm{e}^{-tH_{\mathbb{R}^d}(0,V_n)} \chi_K \right\|_{\infty,\infty} = 0.
\tag{5.12}
$$

To prove (5.12) we use the Cauchy-Schwarz inequality in the Feynman-Kac-Itô formula to obtain

$$
\begin{aligned}
\left\| \left(1 - \chi_{B_R} \right) \mathrm{e}^{-tH_{\mathbb{R}^d}(0,V_n)} \chi_K \right\|_{\infty,\infty} &= \sup_{x\notin B_R} \mathbb{E}_x\left[\mathrm{e}^{-S_t(0,V_n|w)} \chi_K(w(t)) \right] \\
&\leq \left(\left\| \left(1 - \chi_{B_R} \right) \mathrm{e}^{-tH_{\mathbb{R}^d}(0,0)} \chi_K \right\|_{\infty,\infty} \right)^{1/2} \left(\left\| \mathrm{e}^{-tH_{\mathbb{R}^d}(0,2V_n)} \right\|_{\infty,\infty} \right)^{1/2}.
\end{aligned}
\tag{5.13}
$$

By Lemma C.1 the second factor on the right-hand side is uniformly bounded with respect to $n \in \mathbb{N}$ and $0 \leq t \leq \tau_2$. The first factor is seen to vanish uniformly in $0 \leq t \leq \tau_2$ as $R \to \infty$ by an elementary calculation.

As to assertion 3)

In a first step we get with the help of the triangle inequality

$$
\left\| \chi_K \left(\mathrm{e}^{-2tH_\Lambda(A,V)} - \mathrm{e}^{-2tH_\Lambda(A_m,V_n)} \right) \right\|_{1,\infty} \leq N_1 + N_2 + N_3,
\tag{5.14}
$$

where

$$
N_1 := \left\| \chi_K \left(\mathrm{e}^{-tH_\Lambda(A,V)} - \mathrm{e}^{-tH_\Lambda(A_m,V_n)} \right) \mathrm{e}^{-tH_\Lambda(A,V)} \right\|_{1,\infty},
\tag{5.15}
$$

$$
N_2 := \left\| \chi_K \mathrm{e}^{-tH_\Lambda(A_m,V_n)} \chi_{B_R} \left(\mathrm{e}^{-tH_\Lambda(A,V)} - \mathrm{e}^{-tH_\Lambda(A_m,V_n)} \right) \right\|_{1,\infty},
\tag{5.16}
$$

$$
N_3 := \left\| \chi_K \mathrm{e}^{-tH_\Lambda(A_m,V_n)} \left(1 - \chi_{B_R} \right) \left(\mathrm{e}^{-tH_\Lambda(A,V)} - \mathrm{e}^{-tH_\Lambda(A_m,V_n)} \right) \right\|_{1,\infty}.
\tag{5.17}
$$

In a second step we will repeatedly use the inequality

$$
\| XY \|_{p,r} \leq \| X \|_{q,r} \| Y \|_{p,q}
\tag{5.18}
$$

for bounded operators $Y : \mathrm{L}^p(\mathbb{R}^d) \to \mathrm{L}^q(\mathbb{R}^d)$ and $X : \mathrm{L}^q(\mathbb{R}^d) \to \mathrm{L}^r(\mathbb{R}^d)$. By (5.18) and (2.40) we get

$$
N_1 \leq \left\| \chi_K \left(\mathrm{e}^{-tH_\Lambda(A,V)} - \mathrm{e}^{-tH_\Lambda(A_m,V_n)} \right) \right\|_{\infty,\infty} \left\| \mathrm{e}^{-tH_{\mathbb{R}^d}(0,V)} \right\|_{1,\infty}.
\tag{5.19}
$$

By employing additionally the self-adjointness of the semigroup we obtain

$$N_2 \leq \left\| e^{-tH_{\mathbb{R}^d}(0,V_n)} \right\|_{1,\infty} \left\| \left(e^{-tH_\Lambda(A,V)} - e^{-tH_\Lambda(A_m,V_n)} \right) \chi_{B_R} \right\|_{\infty,\infty} \tag{5.20}$$

and similarly

$$\begin{aligned}
N_3 \leq &\left\| \chi_K \, e^{-tH_{\mathbb{R}^d}(0,V_n)} \left(1 - \chi_{B_R} \right) \right\|_{\infty,\infty} \\
&\times \left(\left\| e^{-tH_{\mathbb{R}^d}(0,V)} \right\|_{1,\infty} + \left\| e^{-tH_{\mathbb{R}^d}(0,V_n)} \right\|_{1,\infty} \right).
\end{aligned} \tag{5.21}$$

Hence assertion 3 follows from assertions 1 and 2 together with Lemma C.1 and an asymptotic relation analogous to (5.12). ∎

Remarks 5.2

i) According to (2.29) the analytic condition (5.7) is equivalent to the probabilistic condition

$$\lim_{t \downarrow 0} \sup_{n \in \mathbb{N}} \sup_{x \in \mathbb{R}^d} \mathbb{E}_x \left[\int_0^t V_n^-(w(s))\, \mathrm{d}s \right] = 0. \tag{5.22}$$

ii) Theorem 5.1 is a generalization of [58, Theorem B.10.2] both to non-zero magnetic fields and $\Lambda \subseteq \mathbb{R}^d$. Even for $A = A_m = 0$ and $\Lambda = \mathbb{R}^d$ the result is slightly stronger than that of [58, Theorem B.10.2]. Nevertheless, we followed a similar strategy for the proof.

iii) Propositions 2.3 and 2.6 imply that for given $A \in \mathcal{H}_{\mathrm{loc}}(\mathbb{R}^d)$ and $V \in \mathcal{K}_\pm(\mathbb{R}^d)$ one can find sequences $\{A_m\}_{m \in \mathbb{N}} \subset \left(\mathcal{C}_0^\infty(\mathbb{R}^d) \right)^d$ and $\{V_n\}_{n \in \mathbb{N}} \subset \mathcal{C}_0^\infty(\mathbb{R}^d)$ obeying the hypotheses of Theorem 5.1.

Since the notion of local-norm-continuity occurring in Theorem 5.1 seems to us less common, it may be worth noting that it implies strong continuity of the semigroup in the potentials under the additional condition $p < \infty$.

Corollary 5.3

Let $A \in \mathcal{H}_{loc}(\mathbb{R}^d)$, $\{A_m\}_{m \in \mathbb{N}} \subset \mathcal{H}_{loc}(\mathbb{R}^d)$, $V \in \mathcal{K}_\pm(\mathbb{R}^d)$ and $\{V_n\}_{n \in \mathbb{N}} \subset \mathcal{K}_\pm(\mathbb{R}^d)$ obey (5.4) – (5.7) for all compact $K \subset \mathbb{R}^d$. Moreover, let $\Lambda \subseteq \mathbb{R}^d$ open. Then

$$\lim_{m,n \to \infty} \sup_{\tau_1 \leq t \leq \tau_2} \left\| \left(e^{-tH_\Lambda(A,V)} - e^{-tH_\Lambda(A_m,V_n)} \right) \psi \right\|_q = 0 \tag{5.23}$$

for all $0 < \tau_1 \leq \tau_2 < \infty$, $\psi \in \mathrm{L}^p(\Lambda)$, $1 \leq p < \infty$ being finite, and $1 \leq p \leq q \leq \infty$. Furthermore, for $p = q < \infty$ one may allow $\tau_1 = 0$.

Proof:

We repeatedly use the triangle inequality to achieve the estimate

$$
\begin{aligned}
\left\| \left(\mathrm{e}^{-tH_\Lambda(A,V)} - \mathrm{e}^{-tH_\Lambda(A_m,V_n)} \right) \psi \right\|_q \\
\leq \left\| \left(\mathrm{e}^{-tH_\Lambda(A,V)} - \mathrm{e}^{-tH_\Lambda(A_m,V_n)} \right) \chi_{B_R} \, \psi \right\|_q \\
+ \left(\left\| \mathrm{e}^{-tH_{\mathbb{R}^d}(0,V)} \right\|_{p,q} + \left\| \mathrm{e}^{-tH_{\mathbb{R}^d}(0,V_n)} \right\|_{p,q} \right) \left\| \left(1 - \chi_{B_R} \right) \psi \right\|_p
\end{aligned}
\tag{5.24}
$$

valid for any $R > 0$. The right-hand side of the estimate is uniformly bounded in $n, m \in \mathbb{N}$, $\tau_1 \leq t \leq \tau_2$, due to (2.40) and Lemma C.1, respectively. The proof is therefore accomplished with the help of Theorem 5.1 and the fact that

$$
\lim_{R \to \infty} \left\| \left(1 - \chi_{B_R} \right) \psi \right\|_p = 0,
\tag{5.25}
$$

whenever $\psi \in \mathrm{L}^p(\Lambda)$, $1 \leq p < \infty$. ∎

Remark 5.4

For $p = 2$ Corollary 5.3 is a special case of [42, Theorem 2.8] as can be seen from Remark 2.2.ii).

If one defers norm-continuity in the potentials instead of local-norm-continuity, one has to make stronger assumptions as indicated in the beginning of this section.

Theorem 5.5

Let $\{A_m\}_{m \in \mathbb{N}} \subset \mathcal{H}(\mathbb{R}^d)$ and $\{V_n\}_{n \in \mathbb{N}} \subset \mathcal{K}(\mathbb{R}^d)$ such that

$$
\lim_{m \to \infty} \left\| (A_m)^2 \right\|_{\mathcal{K}(\mathbb{R}^d)} = 0,
\tag{5.26}
$$

$$
\lim_{m \to \infty} \left\| \nabla \cdot A_m \right\|_{\mathcal{K}(\mathbb{R}^d)} = 0
\tag{5.27}
$$

and

$$
\lim_{n \to \infty} \left\| V_n \right\|_{\mathcal{K}(\mathbb{R}^d)} = 0.
\tag{5.28}
$$

Moreover, let $A \in \mathcal{H}_{loc}(\mathbb{R}^d)$, $V \in \mathcal{K}_\pm(\mathbb{R}^d)$ and $\Lambda \subseteq \mathbb{R}^d$ open. Then

$$
\lim_{m,n \to \infty} \sup_{\tau_1 \leq t \leq \tau_2} \left\| \left(\mathrm{e}^{-tH_\Lambda(A,V)} - \mathrm{e}^{-tH_\Lambda(A+A_m,V+V_n)} \right) \right\|_{p,q} = 0
\tag{5.29}
$$

for all $0 < \tau_1 \leq \tau_2 < \infty$ and $1 \leq p \leq q \leq \infty$. Furthermore, for $p = q$ one may allow $\tau_1 = 0$.

Proof:

According to the Riesz-Thorin interpolation theorem [50, Theorem IX.17] and the self-adjointness of the semigroup it is enough to prove (5.29) for the two cases $p = q = \infty$, $\tau_1 = 0$ and $p = 1$, $q = \infty$, $\tau_1 > 0$.

Since the Feynman-Kac-Itô formula (2.34), the triangle inequality and the Cauchy-Schwarz inequality give

$$\left\| e^{-tH_\Lambda(A,V)} - e^{-tH_\Lambda(A+A_m,V+V_n)} \right\|_{\infty,\infty}$$
$$\leq \left(\left\| e^{-tH_{\mathbb{R}^d}(0,2V)} \right\|_{\infty,\infty} \right)^{1/2} \left(\sup_{x \in \mathbb{R}^d} \mathbb{E}_x \left[\left| 1 - e^{-S_t(A_m,V_n|w)} \right|^2 \right] \right)^{1/2}, \tag{5.30}$$

the case $p = q = \infty$, $\tau_1 = 0$ follows from (2.40) and Lemma C.4.

By reasoning in a similar way to the proof of assertion 3 in the proof of Theorem 5.1 we get the estimate

$$\left\| e^{-2tH_\Lambda(A,V)} - e^{-2tH_\Lambda(A+A_m,V+V_n)} \right\|_{1,\infty}$$
$$\leq \left\| e^{-tH_\Lambda(A,V)} - e^{-tH_\Lambda(A+A_m,V+V_n)} \right\|_{\infty,\infty} \tag{5.31}$$
$$\times \left(\left\| e^{-tH_{\mathbb{R}^d}(0,V)} \right\|_{1,\infty} + \left\| e^{-tH_{\mathbb{R}^d}(0,V+V_n)} \right\|_{1,\infty} \right).$$

The second factor on the right-hand side of (5.31) is uniformly bounded with respect to $n \in \mathbb{N}$ and $\tau_1 \leq t \leq \tau_2$ due to (2.40) and Lemma C.1, the latter being applicable because $\{V_n\}_{n \in \mathbb{N}} \subset \mathcal{K}(\mathbb{R}^d)$ and (5.28) together with the definition (2.10) imply (5.7). Thus the case $p = 1$, $q = \infty$, $\tau_1 > 0$ follows with the help of the preceding case. ∎

Remarks 5.6

    i)   Theorem 5.5 is a generalization of [58, Theorem B.10.1].

   ii)   In analogy to Remark 5.2.*iii)* one can find for given $A \in \mathcal{H}_{\text{loc}}(\mathbb{R}^d)$ and $V \in \mathcal{K}_\pm(\mathbb{R}^d)$ sequences $\{A + A_m\}_{m \in \mathbb{N}} \subset \left(\mathcal{C}^\infty(\mathbb{R}^d) \right)^d$ and $\{V + V_n\}_{n \in \mathbb{N}} \subset \mathcal{C}^\infty(\mathbb{R}^d)$ obeying the hypotheses of Theorem 5.5. In contrast to Remark 5.2.*iii)* it is in general wrong to replace here the set $\mathcal{C}^\infty(\mathbb{R}^d)$ of arbitrarily often differentiable functions by $\mathcal{C}_0^\infty(\mathbb{R}^d)$, since the Kato-norm closure of $\mathcal{C}_0^\infty(\mathbb{R}^d)$ is a proper subspace of $\mathcal{K}(\mathbb{R}^d)$, see [68, Proposition 5.5].

6   Continuity of the integral kernel of the semigroup

From the Dunford-Pettis theorem [65, Theorem 46.1], [13, Corollary 2.14] it follows with the help of (2.40) that the operator $e^{-tH_\Lambda(A,V)} : \mathrm{L}^p(\Lambda) \to \mathrm{L}^q(\Lambda)$, $t > 0$, $1 \leq p \leq q \leq \infty$, $A \in \mathcal{H}_{\text{loc}}(\mathbb{R}^d)$, $V \in \mathcal{K}_\pm(\mathbb{R}^d)$, $\Lambda \subseteq \mathbb{R}^d$ open, has an **integral kernel** $k_t : \Lambda \times \Lambda \to \mathbb{C}$ in the sense that

$$\left(e^{-tH_\Lambda(A,V)} \psi \right)(x) = \int_\Lambda k_t(x,y) \, \psi(y) \, \mathrm{d}y \tag{6.1}$$

for all $\psi \in \mathrm{L}^p(\Lambda)$ and almost all $x \in \Lambda$. Furthermore, the integral kernel is bounded according to

$$\operatorname*{ess\,sup}_{x,y \in \Lambda} |k_t(x,y)| = \left\| \mathrm{e}^{-tH_\Lambda(A,V)} \right\|_{1,\infty}. \tag{6.2}$$

The existence of an integral kernel can also be inferred from the Feynman-Kac-Itô formula (2.34) by conditioning the Brownian motion to arrive at y at time t. The resulting representation

$$k_t(x,y) = (2\pi t)^{-d/2} \, \mathrm{e}^{-(x-y)^2/2t} \, \mathbb{E}_x \big[\mathrm{e}^{-S_t(A,V|w)} \, \Xi_{\Lambda,t}(w) \big| \, w(t) = y \big] \tag{6.3}$$

holds for almost all pairs $(x,y) \in \Lambda \times \Lambda$ and all $t > 0$. The purpose of this section is to show that there is a representative of the integral kernel, which is jointly continuous in (t,x,y), $t > 0$, $x,y \in \Lambda$. Moreover, this representative is given in terms of an expectation with respect to the Brownian bridge.

To begin with, we collect some preparatory material concerning the Brownian bridge. We define a continuous version b of the **Brownian bridge** from $x = b(0)$ to $y = b(t)$ in terms of standard Brownian motion w starting from $x = w(0)$ by

$$b(s) := \begin{cases} \dfrac{s}{t} y + w(s) - (t-s) \displaystyle\int_0^s \frac{w(u)}{(t-u)^2} \, \mathrm{d}u & 0 \le s < t \\[2mm] y & s = t \end{cases} \quad \text{for} \quad . \tag{6.4}$$

See [36, Section 5.6.B] or [52, Exercise IX.2.12]. We note that b is adapted to the standard filtration [36, Definition 1.2.25] generated by w and is a continuous semi-martingale [49, Example V.6.3]. The probability measure $\mathbb{P}_{0,x}^{t,y}$ associated with the Brownian bridge is the image of the Wiener measure $\mathbb{P}_x$ by the mapping $w \mapsto b$ as given by (6.4). The induced expectation is written as $\mathbb{E}_{0,x}^{t,y}$.

For any stochastic process $s \mapsto Z(s)$, $0 \le s < t$, adapted to the standard filtration generated by w with $\mathbb{E}_x[|Z(s)|] < \infty$ one has the relation [46, Lemma 3.1]

$$\mathbb{E}_{0,x}^{t,y}[Z(s)] = \left(\frac{t}{t-s} \right)^{d/2} \mathrm{e}^{(x-y)^2/2t} \, \mathbb{E}_x \left[Z(s) \, \mathrm{e}^{-(w(s)-y)^2/2(t-s)} \right] \tag{6.5}$$

for all $0 \le s < t$ and all $x,y \in \mathbb{R}^d$. This is a consequence of the Cameron-Martin-Girsanov theorem [36, Theorem 3.5.1], since the Brownian bridge is obtained from Brownian motion by adding a non-stationary drift, see (6.4). We note that one may alternatively use the right-hand side of (6.5) to construct the Brownian-bridge measure, see [64, Proposition A.1].

For a vector potential A obeying

$$\mathbb{P}_{0,x}^{t,y} \left\{ \int_0^t (A(b(s)))^2 \, \mathrm{d}s < \infty \right\} = 1 \tag{6.6}$$

and

$$\mathbb{P}_{0,x}^{t,y} \left\{ \left| \int_0^t A(b(s)) \cdot \frac{y - b(s)}{t-s} \, \mathrm{d}s \right| < \infty \right\} = 1 \tag{6.7}$$

the stochastic line integral with respect to the Brownian bridge

$$s \mapsto \int_0^s A(b(u)) \cdot \mathrm{d}b(u) := \int_0^s A(b(u)) \cdot \mathrm{d}w(u) + \int_0^s A(b(u)) \cdot \frac{y - b(u)}{t - u} \, \mathrm{d}u \tag{6.8}$$

is a well-defined stochastic process for $0 \le s \le t$ having a continuous version, confer [52, Definition IV.2.9]. According to Lemma C.8, our usual assumption $A \in \mathcal{H}_{\mathrm{loc}}(\mathbb{R}^d)$ suffices to ensure the conditions (6.6) and (6.7).

Now we are able to state the main result of this section:

Theorem 6.1

Let $A \in \mathcal{H}_{loc}(\mathbb{R}^d)$, $V \in \mathcal{K}_{\pm}(\mathbb{R}^d)$, $1 \le p \le q \le \infty$ and $\Lambda \subseteq \mathbb{R}^d$ open. Recall the definitions (2.33) and (2.35) of S_t and $\Xi_{\Lambda,t}$. Then

$$k_t(x, y) := (2\pi t)^{-d/2} \; \mathrm{e}^{-(x-y)^2/2t} \; \mathbb{E}_{0,x}^{t,y} \left[\mathrm{e}^{-S_t(A,V|b)} \, \Xi_{\Lambda,t}(b) \right] \tag{6.9}$$

defines pointwise that integral kernel for

$$\mathrm{e}^{-tH_\Lambda(A,V)} : \mathrm{L}^p(\Lambda) \to \mathrm{L}^q(\Lambda) \tag{6.10}$$

which is jointly continuous in (t, x, y), $t > 0$, $x, y \in \Lambda$.

Proof:

Since the Brownian-bridge expectation in (6.9) yields a regular version of the conditional expectation in (6.3), k_t as defined by (6.9) is an integral kernel for (6.10). Moreover, recalling our preparations, the kernel k_t is well defined for all $t > 0$, $x, y \in \Lambda$. It remains to show the claimed continuity of the function k defined by (6.9).

Employing the time-reversal symmetry of the Brownian bridge we get from (6.9) the Hermiticity of the integral kernel, that is, $k_t(x, y)$ turns into its complex conjugate upon interchanging x and y. Therefore, it is sufficient to ensure

$$\lim_{\varrho \downarrow 0} \sup_{\tau_1 \le t \le t' \le \tau_2, |t-t'| < \varrho} \sup_{y,y' \in K, |y-y'| < \varrho} \sup_{x \in K} |k_t(x, y) - k_{t'}(x, y')| = 0 \tag{6.11}$$

for all compact $K \subset \Lambda$ and all $0 < \tau_1 \le \tau_2 < \infty$. Note that we have assumed $t \le t'$.

For any $0 < s < \tau_1$ we can rewrite the difference in (6.11) according to

$$k_t(x, y) - k_{t'}(x, y') = \Upsilon(t', t' - t + s, x, y') - \Upsilon(t, s, x, y) + \Gamma(t, t', s, x, y, y'), \tag{6.12}$$

where we have introduced the abbreviations

$$\begin{aligned} \Upsilon(t, s, x, y) := \; & (2\pi t)^{-d/2} \; \mathrm{e}^{-(x-y)^2/2t} \\ & \times \mathbb{E}_{0,x}^{t,y} \left[\mathrm{e}^{-S_t(A,V|b)} \, \Xi_{\Lambda,t}(b) - \mathrm{e}^{-S_{t-s}(A,V|b)} \, \Xi_{\Lambda,t-s}(b) \right] \end{aligned} \tag{6.13}$$

and

$$\begin{aligned} \Gamma(t, t', s, x, y, y') := \; & (2\pi t')^{-d/2} \; \mathrm{e}^{-(x-y')^2/2t'} \, \mathbb{E}_{0,x}^{t',y'} \left[\mathrm{e}^{-S_{t-s}(A,V|b)} \, \Xi_{\Lambda,t-s}(b) \right] \\ & - (2\pi t)^{-d/2} \; \mathrm{e}^{-(x-y)^2/2t} \, \mathbb{E}_{0,x}^{t,y} \left[\mathrm{e}^{-S_{t-s}(A,V|b)} \, \Xi_{\Lambda,t-s}(b) \right]. \end{aligned} \tag{6.14}$$

Therefore, it is enough to show

$$\lim_{s \downarrow 0} \sup_{\tau_1 \le t \le \tau_2} \sup_{x,y \in K} |\Upsilon(t, s, x, y)| = 0 \tag{6.15}$$

and

$$\lim_{\varrho \downarrow 0} \sup_{\tau_1 \le t \le t' \le \tau_2, |t-t'|<\varrho} \sup_{y,y' \in \mathbb{R}^d, |y-y'|<\varrho} \sup_{x \in \mathbb{R}^d} |\Gamma(t, t', s, x, y, y')| = 0 \tag{6.16}$$

for all compact $K \subset \Lambda$ and all $0 < s < \tau_1 \le \tau_2 < \infty$. The assertion (6.16) is actually stronger than needed for the present purpose, but will turn out to be useful in the sequel.

From the triangle inequality we get

$$\begin{aligned}
\Upsilon(t, s, x, y) &\le (2\pi t)^{-d/2} \, \mathrm{e}^{-(x-y)^2/2t} \\
&\times \Big(\mathbb{E}_{0,x}^{t,y} \big[\big| \mathrm{e}^{-S_t(A,V|b)} - \mathrm{e}^{-S_{t-s}(A,V|b)} \big| \big] \\
&\quad + \mathbb{E}_{0,x}^{t,y} \big[\mathrm{e}^{-S_{t-s}(0,V|b)} \, | \Xi_{\Lambda,t}(b) - \Xi_{\Lambda,t-s}(b) | \big] \Big).
\end{aligned} \tag{6.17}$$

Now we make use of the Cauchy-Schwarz inequality, the elementary estimate

$$S_{t-s}(0, 2V|b) \ge S_t(0, -2V^-|b) \tag{6.18}$$

and the time-reversal symmetry of the Brownian bridge. We thus achieve

$$\begin{aligned}
\Upsilon(t, s, x, y) &\le N^{1/2} \, (2\pi t)^{-d/4} \, \mathrm{e}^{-(x-y)^2/4t} \\
&\times \Big(\big(\mathbb{E}_{0,y}^{t,x} \big[\big| 1 - \mathrm{e}^{-S_s(A,V|b)} \big|^2 \big] \big)^{1/2} + \big(\mathbb{E}_{0,y}^{t,x} [\, 1 - \Xi_{\Lambda,s}(b)] \big)^{1/2} \Big) \\
&\le N^{1/2} \, (2\pi(t-s))^{-d/4} \\
&\times \Big(\big(\mathbb{E}_y \big[\big| 1 - \mathrm{e}^{-S_s(A,V|w)} \big|^2 \big] \big)^{1/2} + \big(\mathbb{E}_y [\, 1 - \Xi_{\Lambda,s}(w)] \big)^{1/2} \Big).
\end{aligned} \tag{6.19}$$

Here we have used (6.5) for the second step and have set

$$N := \sup_{\tau_1 \le t \le \tau_2} \sup_{x,y \in \mathbb{R}^d} (2\pi t)^{-d/2} \, \mathrm{e}^{-(x-y)^2/2t} \, \mathbb{E}_{0,x}^{t,y} \Big[\mathrm{e}^{-S_t(0,-2V^-|b)} \Big]. \tag{6.20}$$

Employing the Cauchy-Schwarz inequality, the time-reversal symmetry and (6.5) again, we get

$$N \le (\pi \tau_1)^{-d/2} \sup_{x \in \mathbb{R}^d} \mathbb{E}_x \Big[\mathrm{e}^{-S_{\tau_2/2}(0,-4V^-|w)} \Big], \tag{6.21}$$

which shows by virtue of Lemma C.2 that N is finite. Note that it is essential to take the supremum in (6.20) and not only the essential supremum as, for example, in (6.2). Having shown that N is finite, (6.15) follows from (6.19) with the help of (3.11) and Lemma C.5.

The remaining task is to prove (6.16). In a first step we use (6.5) to rewrite Γ as given by (6.14) in terms of an expectation with respect to Brownian motion. Then the Cauchy-Schwarz inequality and (6.18) lead to

$$\begin{aligned}
| \Gamma(t, t', s, x, y, y') |^2 &\le (2\pi)^{-d} M \, \mathbb{E}_x \Big[\Big((t' - t + s)^{-d/2} \, \mathrm{e}^{-(w(t-s)-y')^2/2(t'-t+s)} \\
&\qquad - s^{-d/2} \, \mathrm{e}^{-(w(t-s)-y)^2/2s} \Big)^2 \Big],
\end{aligned} \tag{6.22}$$

where

$$M := \sup_{x \in \mathbb{R}^d} \mathbb{E}_x \left[e^{-S_{\tau_2}(0, -2V^-|w)} \right] \tag{6.23}$$

is finite due to Lemma C.2. The expectation on the right-hand side of (6.22) can be calculated explicitly. The result implies

$$
\begin{aligned}
| \Gamma(t, t', s, x, y, y') |^2 & \\
\leq (2\pi)^{-d} M & \left[\left(t'^2 - (t-s)^2 \right)^{-d/2} \exp\left\{ -\frac{(x-y')^2}{t'+t-s} \right\} \right. \\
& + s^{-d/2} (2t-s)^{-d/2} \exp\left\{ -\frac{(x-y)^2}{2t-s} \right\} \\
& - 2 \left(t\,t' - (t-s)^2 \right)^{-d/2} \\
& \left. \times \exp\left\{ -\frac{s(x-y')^2 + (t-s)(y-y')^2 + (t'-t+s)(x-y)^2}{2(t\,t' - (t-s)^2)} \right\} \right].
\end{aligned}
\tag{6.24}
$$

Now we use the three elementary inequalities

$$\frac{1}{t'^2 - (t-s)^2} \leq \frac{1}{s(2t-s)}, \quad \frac{1}{2t-s} \geq \frac{1}{t'+t-s}, \quad \frac{1}{t\,t' - (t-s)^2} \leq \frac{1}{s(t'+t-s)}, \tag{6.25}$$

valid for all $0 < s < t \leq t' < \infty$, and get

$$
\begin{aligned}
| \Gamma(t, t', s, x, y, y') |^2 & \\
\leq 2(2\pi)^{-d} M\, s^{-d/2} (2t-s)^{-d/2} & \exp\left\{ -\frac{1}{2} \frac{(x-y')^2 + (x-y)^2}{t'+t-s} \right\} \\
\times & \left[\cosh\left\{ \frac{(y-y') \cdot (y+y'-2x)}{2(t+t'-s)} \right\} \right. \\
& \left. - \left(1 + \frac{t(t'-t)}{s(2t-s)} \right)^{-d/2} \exp\left\{ -\frac{(t-s)(y-y')^2 + (t'-t)(x-y)^2}{2s(t+t'-s)} \right\} \right].
\end{aligned}
\tag{6.26}
$$

In a final step we further estimate the right-hand side with the help of the elementary inequalities

$$
\begin{aligned}
\cosh(a) &\leq \frac{1}{2} a^2 \cosh(a) + 1, \quad a \in \mathbb{R}, \\
(1+b)^{-d/2} &\geq 1 - \frac{d}{2} b, \quad b > 0, \qquad e^{-c} \geq 1 - c, \quad c \in \mathbb{R},
\end{aligned}
\tag{6.27}
$$

the first one of which may readily be derived from the Taylor expansion [1, Equation 4.5.63] of the hyperbolic cosine. The assertion (6.16) then follows from the resulting bound by inspection. ∎

Remarks 6.2

    i) Theorem 6.1 generalizes [58, Theorem B.7.1.(a‴)] to non-zero $A \in \mathcal{H}_{\mathrm{loc}}(\mathbb{R}^d)$ and $\Lambda \subseteq \mathbb{R}^d$. The proof given there relies on the local-norm-continuity of

the semigroup in the scalar potential and the approximability Proposition 2.3 (see [58, Theorem B.10.2], but beware of a circularity in the proofs of [58, Proposition B.6.7, Theorem B.7.1.(a′), Lemma B.7.4]). By using Theorem 5.1 local-norm-continuity covers also the case $A \neq 0$.

Results similar to Theorem 6.1 for $A = 0$ and $\Lambda \subseteq \mathbb{R}^d$ are contained in [11, Theorem 3.17] and [64, Proposition 1.3.5].

Like the last reference our approach has the advantage of singling out the continuous representative of the integral kernel as a Brownian-bridge expectation, which is useful to know for further investigations and applications.

The rather general Theorem 14.5 in [56] does not cover the above Theorem 6.1. A discriminating example corresponds to the hydrogen atom: $\Lambda = \mathbb{R}^3$, $A = 0$, $V(x) = -|x|^{-1}$. In the notation of [56] this means: $\nu = 3$, $f = 0$, $F(a) = \mathrm{e}^{ia}$ for $a \in \mathbb{R}$, $g(\boldsymbol{\omega}) = \exp\left\{ \int_0^t |\boldsymbol{\omega}(s)|^{-1}\mathrm{d}s \right\}$ for $t > 0$. Theorem 14.5 in [56] is not applicable, because the Wiener-essential supremum $\|g\|_\infty$ of g is infinite.

  ii) The continuity of the integral kernel of the semigroup implies the continuity of the integral kernels for various functions of the Schrödinger operator $H_\Lambda(A, V)$. This can be shown along the same lines of reasoning as in the proof of [58, Theorem B.7.1], since the arguments given there do not use the fact that $H_{\mathbb{R}^d}(0, V)$ is a particular Schrödinger operator, but merely the continuity of $k_t(x, y)$ and the bound (2.40). For example, each spectral projection

$$\chi_\Omega(H_\Lambda(A, V)) : \mathrm{L}^2(\Lambda) \to \mathrm{L}^2(\Lambda) \tag{6.28}$$

associated with a bounded Borel subset $\Omega \subset \mathbb{R}$ is an integral operator with jointly continuous kernel.

  iii) Since $k_t(x, y)$ is jointly continuous in (x, y) for $t > 0$, the trace of the non-negative operator $\mathrm{e}^{-tH_\Lambda(A,V)} : \mathrm{L}^2(\Lambda) \to \mathrm{L}^2(\Lambda)$ can be represented as

$$\mathrm{tr}\, \mathrm{e}^{-tH_\Lambda(A,V)} = \int_\Lambda k_t(x, x)\, \mathrm{d}x, \tag{6.29}$$

whenever one of the two sides of this equation is finite.

  iv) Due to the continuity of the integral kernel, the diamagnetic inequality (2.38) takes the form

$$|k_t(x, y)| \leq k_t(x, y)|_{A=0} \tag{6.30}$$

and is valid pointwise for all $x, y \in \Lambda$, $t > 0$. Various upper bounds on the right-hand side of (6.30) are available in the literature, see, for example, the estimate [58, Theorem B.7.1.(a′)]. Using this estimate, Theorem 4.1 can be proven alternatively with the help of the continuity of the integral kernel and the dominated-convergence theorem. We have given a different proof of Theorem 4.1, mainly because we think it is nice to see how Carmona's elegant argument for $A = 0$ can be extended to $A \neq 0$.

Theorem 6.3

Let $A \in \mathcal{H}_{loc}(\mathbb{R}^d)$, $V \in \mathcal{K}_{\pm}(\mathbb{R}^d)$ and $\Lambda \subseteq \mathbb{R}^d$ regular. Then $k_t(x, y)$, as defined by (6.9), is jointly continuous in (t, x, y), $t > 0$, $x, y \in \overline{\Lambda}$ and vanishes if $x \in \partial\Lambda$ or $y \in \partial\Lambda$.

Proof:

From (6.9) we derive with the help of the Cauchy-Schwarz inequality and $\Xi_{\Lambda,t}(b) \leq \Xi_{\Lambda,t/2}(b)$ the estimate

$$|k_t(x, y)| \leq (2\pi t)^{-d/4} \, N^{1/2} \, e^{-(x-y)^2/4t} \, \left(\mathbb{E}_{0,x}^{t,y}\left[\Xi_{\Lambda,t/2}(b) \right] \right)^{1/2}, \tag{6.31}$$

where $0 \leq N < \infty$ is given by (6.20). With the help of (6.5) we achieve

$$|k_t(x, y)| \leq (\pi t)^{-d/4} \, N^{1/2} \, \left(\mathbb{E}_x\left[\Xi_{\Lambda,t/2}(b) \right] \right)^{1/2}. \tag{6.32}$$

According to Lemma C.7 the right-hand side vanishes as x approaches any given point on the boundary $\partial\Lambda$. Due to the Hermiticity of k_t and Theorem 6.1 the assertion follows. ∎

Remark 6.4

Theorem 6.3 is a partial generalization of [11, Theorem 3.17] to $A \neq 0$.

We finally want to show that the same strategy as used in Section 4 allows to get uniform continuity for global regularity conditions.

Theorem 6.5

Let $A \in \mathcal{H}(\mathbb{R}^d)$, $V \in \mathcal{K}(\mathbb{R}^d)$, $0 < \tau_1 < \tau_2 < \infty$ and $\Lambda \subseteq \mathbb{R}^d$ uniformly regular. Then the function

$$[\tau_1, \tau_2] \times \overline{\Lambda} \times \overline{\Lambda} \to \mathbb{C}, \quad (t, x, y) \mapsto k_t(x, y) \tag{6.33}$$

is uniformly continuous.

Proof:

Analogously to the reasoning at the beginning of the proof of Theorem 6.1, it is sufficient to ensure

$$\lim_{\varrho \downarrow 0} \sup_{\tau_1 \leq t \leq t' \leq \tau_2, |t-t'| < \varrho} \sup_{y,y' \in \Lambda_r, |y-y'| < \varrho} \sup_{x \in \Lambda_r} |k_t(x, y) - k_{t'}(x, y')| = 0 \tag{6.34}$$

in order to get uniform continuity on $[\tau_1, \tau_2] \times \Lambda_r \times \Lambda_r$, $r > 0$, where Λ_r is given by (3.12). Due to (6.12) and (6.16) we only have to check that

$$\lim_{s \downarrow 0} \sup_{\tau_1 \leq t \leq \tau_2} \sup_{x,y \in \Lambda_r} |\Upsilon(t, s, x, y)| = 0, \tag{6.35}$$

where Υ is given by (6.13). With the help of the estimate (6.19) this follows from Lemma C.3 and (3.11).

On the other hand, Lemma C.7 and (6.32) imply

$$\lim_{r\downarrow 0} \sup_{\tau_1\le t\le\tau_2} \sup_{x\in\Lambda_r} \sup_{y\in\Lambda\setminus\Lambda_r} |k_t(x,y)| = 0, \tag{6.36}$$

which suffices to extend the domain of uniform continuity from $[\tau_1,\tau_2]\times\Lambda_r\times\Lambda_r$, $r>0$, to $[\tau_1,\tau_2]\times\overline{\Lambda}\times\overline{\Lambda}$. ■

Appendix A   Approximability of Kato-type potentials by smooth functions

We want to prove Propositions 2.3 and 2.6. The idea of the proof has been given already in Section 2. It is similar to the one underlying the standard proof of the fact that $\mathcal{C}_0^\infty(\mathbb{R}^d)$ is dense in $\mathrm{L}^p(\mathbb{R}^d)$ for $1\le p<\infty$.

To begin with, we choose $\delta_1\in\mathcal{C}_0^\infty(\mathbb{R}^d)$ such that $\delta_1\ge 0$, $\delta_1(x)=0$ for $|x|>1$ and $\|\delta_1\|_1=1$. Moreover, we set $\delta_r(x):=r^{-d}\delta_1(x/r)$ for $0<r\le 1$ and $\Theta_R:=\delta_1*\chi_{B_R}$ for $R>1$, where

$$(h*F)(x) := \int h(x-y)\,F(y)\,\mathrm{d}y \tag{A.1}$$

denotes the convolution of a function $F:\mathbb{R}^d\to\mathbb{R}^\nu$, $\nu\in\mathbb{N}$, with a function $h:\mathbb{R}^d\to\mathbb{R}$. We note that $\Theta_R\in\mathcal{C}_0^\infty(\mathbb{R}^d)$ and

$$\chi_{B_{R-1}} \le \Theta_R \le \chi_{B_{R+1}}. \tag{A.2}$$

Now we are ready to formulate the following preparatory result.

Lemma A.1

Let $F:\mathbb{R}^d\to\mathbb{R}^\nu$, $|F|^p\in\mathcal{K}_{loc}(\mathbb{R}^d)$, $1\le p<\infty$. Then

$$\delta_r*\Theta_R F \in \left(\mathcal{C}_0^\infty(\mathbb{R}^d)\right)^\nu, \tag{A.3}$$

$$\left\|g_\varrho*|\delta_r*\Theta_R F|^p\right\|_\infty \le \left\|g_\varrho*\chi_{B_{R+1}}|F|^p\right\|_\infty \tag{A.4}$$

and

$$\lim_{r\downarrow 0}\left\||\Theta_R F - \delta_r*\Theta_R F|^p\right\|_{\mathcal{K}(\mathbb{R}^d)} = 0 \tag{A.5}$$

for all $0<r\le 1<R<\infty$ and $0<\varrho\le 1$.

Proof:

(A.3) holds, since $\Theta_R F\in\left(\mathrm{L}^1(\mathbb{R}^d)\right)^\nu$ by (2.12).

From the Jensen inequality and (A.2) we have

$$|\delta_r*\Theta_R F|^p \le \delta_r*\chi_{B_{R+1}}|F|^p. \tag{A.6}$$

Furthermore, by the commutativity and the associativity of the convolution and the monotonicity of integration we get

$$g_\varrho * \delta_r * \chi_{B_{R+1}} |F|^p \le \|\delta_r\|_1 \left\| g_\varrho * \chi_{B_{R+1}} |F|^p \right\|_\infty. \tag{A.7}$$

Now (A.4) follows from (A.6), (A.7) and $\|\delta_r\|_1 = 1$.

Since $\|f\|_{\mathcal{K}(\mathbb{R}^d)} = \|g_1 * |f|\|_\infty$, we observe for $f \in \mathcal{K}(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$ that

$$\|f\|_{\mathcal{K}(\mathbb{R}^d)} \le \|g_\varrho * |f|\|_\infty + G_\varrho \|f\|_1, \tag{A.8}$$

where

$$G_\varrho := \sup_{\varrho \le |x| \le 1} g_1(x). \tag{A.9}$$

On the other hand, the inequality $|u - v|^p \le 2^p (|u|^p + |v|^p)$ for $u, v \in \mathbb{R}^\nu$ in combination with (A.2) and (A.6) yields

$$|\Theta_R F - \delta_r * \Theta_R F|^p \le 2^p \left(\chi_{B_{R+1}} |F|^p + \delta_r * \chi_{B_{R+1}} |F|^p \right). \tag{A.10}$$

With the help of (A.8), (A.10) and (A.4) we establish

$$\begin{aligned} &\left\| |\Theta_R F - \delta_r * \Theta_R F|^p \right\|_{\mathcal{K}(\mathbb{R}^d)} \\ &\le 2^{p+1} \left\| g_\varrho * \chi_{B_{R+1}} |F|^p \right\|_\infty + G_\varrho \left\| |\Theta_R F - \delta_r * \Theta_R F|^p \right\|_1. \end{aligned} \tag{A.11}$$

Since $\Theta_R F \in (L^p(\mathbb{R}^d))^\nu$, the second term on the right-hand side tends to zero as $r \downarrow 0$, see, for example, [66, Lemma 1.3.6/2]. Moreover, the remaining first term vanishes as $\varrho \downarrow 0$ because $\chi_{B_{R+1}} |F|^p \in \mathcal{K}(\mathbb{R}^d)$. This proves (A.5). ∎

Proof of Proposition 2.3:

By the triangle inequality

$$|V - \delta_r * \Theta_R V| \chi_K \le V (1 - \Theta_R) \chi_K + |\Theta_R V - \delta_r * \Theta_R V| \tag{A.12}$$

and the fact that

$$(1 - \Theta_R) \chi_K = 0 \quad \text{for} \quad R > 1 + \sup_{x \in K} |x| \tag{A.13}$$

the approximability (2.15) follows from (A.3) and (A.5). The inequality (2.16) follows from (A.4) for the choice $p = 1$ and $F = V^-$. ∎

Proof of Proposition 2.6:

We start from the pair of elementary inequalities

$$(A - \delta_r * \Theta_R A)^2 \chi_K \le 2A^2 (1 - \Theta_R) \chi_K + 2 (\Theta_R A - \delta_r * \Theta_R A)^2 \tag{A.14}$$

and

$$|\nabla \cdot A - \nabla \cdot (\delta_r * \Theta_R A)| \, \chi_K$$
$$\leq |\nabla \cdot A| \, (1 - \Theta_R) \, \chi_K + |\delta_r * (A \cdot \nabla \Theta_R)| \, \chi_K \qquad \text{(A.15)}$$
$$+ |\Theta_R \nabla \cdot A - \delta_r * (\Theta_R \nabla \cdot A)| \, .$$

The second term on the right-hand side of (A.15) can be estimated further with the help of (A.2) according to

$$|\delta_r * (A \cdot \nabla \Theta_R)| \, \chi_K \leq \chi_K \, \chi_{B_{R+2}} \left(1 - \chi_{B_{R-2}}\right) (\delta_r * |A|) \sup_{x \in \mathbb{R}^d} |(\nabla \Theta_R)(x)| \qquad \text{(A.16)}$$

and, hence, vanishes for $R > 2 + \sup_{x \in K} |x|$. In consequence, the approximability (2.19), (2.20) follows from (A.3) with the help of (A.13) and (A.5). ∎

Appendix B   Proof of the Feynman-Kac-Itô formula

The purpose of this appendix is to prove Proposition 2.9. The strategy of the proof has already been sketched in Section 2. Here we only recall that we take (2.34) in the case $\Lambda = \mathbb{R}^d$, $A \in \mathcal{H}_{\text{loc}}(\mathbb{R}^d)$ and $V \in \mathcal{K}_{\pm}(\mathbb{R}^d)$ for granted.

Let $\{K_l\}_{l \in \mathbb{N}}$ be a strictly increasing sequence of compact subsets of Λ that exhausts Λ. That is, each K_l is contained in the interior of K_{l+1} and $\bigcup_{l \in \mathbb{N}} K_l = \Lambda$. Furthermore let $\{\vartheta_l\}_{l \in \mathbb{N}} \subset \mathcal{C}_0^\infty(\mathbb{R}^d)$ be such that $\vartheta_l(x) = 1$ for $x \in K_l$, $\vartheta_l(x) = 0$ for $x \notin K_{l+1}$, and $0 \leq \vartheta_l(x) \leq 1$ for all $x \in \mathbb{R}^d$. Define $U_\infty^\Lambda : \mathbb{R}^d \to \mathbb{R}$ by

$$U_\infty^\Lambda(x) := \begin{cases} \sum_{l \in \mathbb{N}} |(\nabla \vartheta_l)(x)|^2 + (\inf\{|x - y| : y \notin \Lambda\})^{-3} & \text{for } x \in \Lambda \\ +\infty & \text{otherwise} \end{cases} \qquad \text{(B.1)}$$

and $U_n^\Lambda : \mathbb{R}^d \to \mathbb{R}$ for $n \in \mathbb{N}$ by

$$U_n^\Lambda(x) := \inf\{U_\infty^\Lambda(x), n\}. \qquad \text{(B.2)}$$

The sesquilinear form $h_{\mathbb{R}^d}^{A, \mu U_\infty^\Lambda + V}$, $A \in \mathcal{H}_{\text{loc}}(\mathbb{R}^d)$, $V \in \mathrm{L}^\infty(\mathbb{R}^d)$, $\mu > 0$, defined through (2.22), with domain $\mathcal{Q}\left(h_{\mathbb{R}^d}^{A, \mu U_\infty^\Lambda + V}\right) = \mathcal{Q}\left(h_{\mathbb{R}^d}^{A, U_\infty^\Lambda}\right)$ given by (2.23) is closed but not densely defined. In fact $\mathcal{C}_0^\infty(\Lambda) \subset \mathcal{Q}\left(h_{\mathbb{R}^d}^{A, U_\infty^\Lambda}\right) \subset \mathrm{L}^2(\Lambda)$ by the extension convention of Section 2 and the form is densely defined on $\mathrm{L}^2(\Lambda)$. It gives rise to a self-adjoint non-negative operator on $\mathrm{L}^2(\Lambda)$ which we denote by $H_{\mathbb{R}^d}(A, \mu U_\infty^\Lambda + V)$. For a function

$$f : \mathbb{R} \to \mathbb{R}, \quad f \text{ continuous}, \quad \lim_{x \to +\infty} f(x) = 0 \qquad \text{(B.3)}$$

we define for $\psi \in \mathrm{L}^2(\mathbb{R}^d)$

$$f\left(H_{\mathbb{R}^d}(A, \mu U_\infty^\Lambda + V)\right) \psi := f\left(H_{\mathbb{R}^d}(A, \mu U_\infty^\Lambda + V)\right) (\chi_\Lambda \, \psi). \qquad \text{(B.4)}$$

Hence $f\left(H_{\mathbb{R}^d}(A, \mu U_\infty^\Lambda + V)\right)$ naturally extends to a bounded self-adjoint operator on $\mathrm{L}^2(\mathbb{R}^d)$.

Our plan for the proof of Proposition 2.9 is summarized in the following six lemmas.

Lemma B.1

Let $A \in \mathcal{H}_{loc}(\mathbb{R}^d)$, $V \in \mathrm{L}^\infty(\mathbb{R}^d)$, $V \geq 0$, $\psi \in \mathrm{L}^2(\mathbb{R}^d)$, $\mu > 0$, $t \geq 0$, $\Lambda \subseteq \mathbb{R}^d$ open. Then

$$\lim_{n \to \infty} \left\| \left(\mathrm{e}^{-tH_{\mathbb{R}^d}(A,\, \mu U_n^\Lambda + V)} - \mathrm{e}^{-tH_{\mathbb{R}^d}(A,\, \mu U_\infty^\Lambda + V)} \right) \psi \right\|_2 = 0. \tag{B.5}$$

Lemma B.1 serves for the proof of the Feynman-Kac-Itô formula for $\mathrm{e}^{-tH_{\mathbb{R}^d}(A,\, \mu U_\infty^\Lambda + V)}$.

Lemma B.2

Let $A \in \mathcal{H}_{loc}(\mathbb{R}^d)$, $V \in \mathrm{L}^\infty(\mathbb{R}^d)$, $V \geq 0$, $\psi \in \mathrm{L}^2(\mathbb{R}^d)$, $\mu > 0$, $t \geq 0$, $\Lambda \subseteq \mathbb{R}^d$ open. Then

$$\left(\mathrm{e}^{-tH_{\mathbb{R}^d}(A,\, \mu U_\infty^\Lambda + V)} \psi \right)(x) = \mathbb{E}_x \left[\mathrm{e}^{-S_t(A,\, \mu U_\infty^\Lambda + V | w)} \Xi_{\Lambda, t}(w)\, \psi(w(t)) \right] \tag{B.6}$$

for almost all $x \in \mathbb{R}^d$.

The purpose of Lemma B.2 is twofold. First, it immediately implies the inequality

$$\left| \mathrm{e}^{-tH_{\mathbb{R}^d}(A,\, \mu U_\infty^\Lambda + V)} \psi \right| \leq \mathrm{e}^{-tH_{\mathbb{R}^d}(0,0)} |\psi|, \quad \psi \in \mathrm{L}^2(\Lambda),\ t \geq 0, \tag{B.7}$$

which is crucial in the proof of the following lemma.

Lemma B.3

Let $A \in \mathcal{H}_{loc}(\mathbb{R}^d)$, $\Lambda \subseteq \mathbb{R}^d$ open. Then the completion of the form domain $\mathcal{Q}\left(h_{\mathbb{R}^d}^{A, U_\infty^\Lambda} \right)$ with respect to the form-norm $\| \bullet \|_{h_{\mathbb{R}^d}^{A,0}}$ is $\mathcal{Q}\left(h_\Lambda^{A,0} \right)$.

This rather technical lemma singles out the difficult part of the proof of the next approximation.

Lemma B.4

Let $A \in \mathcal{H}_{loc}(\mathbb{R}^d)$, $V \in \mathrm{L}^\infty(\mathbb{R}^d)$, $V \geq 0$, $\psi \in \mathrm{L}^2(\mathbb{R}^d)$, $\Lambda \subseteq \mathbb{R}^d$ open, $t \geq 0$. Then

$$\lim_{\mu \downarrow 0} \left\| \left(\mathrm{e}^{-tH_{\mathbb{R}^d}(A,\, \mu U_\infty^\Lambda + V)} - \mathrm{e}^{-tH_\Lambda(A,V)} \right) \psi \right\|_2 = 0. \tag{B.8}$$

The second purpose of Lemma B.2 is, of course, to be used together with B.4 to get the next variant of the Feynman-Kac-Itô formula. Assume $A \in \mathcal{H}_{\mathrm{loc}}(\mathbb{R}^d)$, $V \in \mathrm{L}^\infty(\mathbb{R}^d)$, $V \geq 0$. Then Lemma B.4 implies that the left-hand side of (B.6) tends to the left-hand side of (2.34) for almost all $x \in \Lambda$ as $\mu \downarrow 0$ at least for a suitably chosen subsequence. Furthermore, the right-hand side of (B.6) converges to the right-hand side of (2.34) for almost all $x \in \Lambda$ as $\mu \downarrow 0$ by the dominated-convergence theorem. One may relax the assumption $V \in \mathrm{L}^\infty(\mathbb{R}^d)$, $V \geq 0$ to $V \in \mathrm{L}^\infty(\mathbb{R}^d)$ by multiplying both sides of (2.34) with $\mathrm{e}^{t \inf_{x \in \mathbb{R}^d} V(x)}$. We summarize these findings in the following lemma.

Lemma B.5

Let $A \in \mathcal{H}_{loc}(\mathbb{R}^d)$, $V \in \mathrm{L}^\infty(\mathbb{R}^d)$, $\psi \in \mathrm{L}^2(\Lambda)$, $\Lambda \subseteq \mathbb{R}^d$ open, $t \geq 0$. Then (2.34) holds for almost all $x \in \Lambda$.

With this statement at hand, the diamagnetic inequality (2.25) follows from the triangle inequality. This is just-in-time because we used (2.25) in the case $\Lambda \subset \mathbb{R}^d$, $\Lambda \neq \mathbb{R}^d$ for the construction of $H_\Lambda(A,V)$, $A \in \mathcal{H}_{loc}(\mathbb{R}^d)$, $V \in \mathcal{K}_\pm(\mathbb{R}^d)$, in that case.

Now we are ready to formulate the last approximation tool for the proof of Proposition 2.9.

Lemma B.6

Let $A \in \mathcal{H}_{loc}(\mathbb{R}^d)$, $V \in \mathcal{K}_\pm(\mathbb{R}^d)$, $\psi \in \mathrm{L}^2(\mathbb{R}^d)$, $\Lambda \subseteq \mathbb{R}^d$ open, $t \geq 0$ and set for $n, m \in \mathbb{N}$

$$V_m^n(x) := \sup\{-n, \inf\{m, V(x)\}\} \tag{B.9}$$

and consequently $V_m^\infty(x) = \inf\{m, V(x)\}$. Then

$$\lim_{n \to \infty} \left\| \left(\mathrm{e}^{-tH_\Lambda(A,V_n^m)} - \mathrm{e}^{-tH_\Lambda(A,V_\infty^m)} \right) \psi \right\|_2 = 0 \tag{B.10}$$

and

$$\lim_{m \to \infty} \left\| \left(\mathrm{e}^{-tH_\Lambda(A,V_\infty^m)} - \mathrm{e}^{-tH_\Lambda(A,V)} \right) \psi \right\|_2 = 0. \tag{B.11}$$

Proof of Proposition 2.9

From (B.10) and (B.11) we conclude that for a suitably chosen subsequence

$$\left(\mathrm{e}^{-tH_\Lambda(A,V_n^m)} \psi \right)(x) \to \left(\mathrm{e}^{-tH_\Lambda(A,V)} \psi \right)(x) \tag{B.12}$$

as $n, m \to \infty$ for almost all $x \in \Lambda$. From Lemma B.5 we know that the Feynman-Kac-Itô formula holds for the left-hand side of (B.12) and it therefore remains to show

$$\lim_{n,m \to \infty} \mathbb{E}_x\left[\mathrm{e}^{-S_t(A,V_n^m|w)} \Xi_{\Lambda,t}(w)\, \psi(w(t)) \right] = \mathbb{E}_x\left[\mathrm{e}^{-S_t(A,V|w)} \Xi_{\Lambda,t}(w)\, \psi(w(t)) \right] \tag{B.13}$$

for almost all $x \in \Lambda$. To this end note that

$$\left| \mathrm{e}^{-S_t(A,V_n^m|w)} \Xi_{\Lambda,t}(w) \right| \leq \left| \mathrm{e}^{-S_t(0,V^-|w)} \right| \tag{B.14}$$

and

$$\mathbb{E}_x\left[\mathrm{e}^{-S_t(0,V^-|w)}\, |\psi(w(t))| \right] \leq \left\| \mathrm{e}^{-tH_{\mathbb{R}^d}(0,V^-)} \right\|_{2,\infty} < \infty \tag{B.15}$$

by (2.34) for $\Lambda = \mathbb{R}^d$ and [58, Theorem B.1.1], see also Lemma C.1. Employing the dominated-convergence theorem we conclude that it is enough to establish

$$\lim_{n,m \to \infty} \int_0^t V_n^m(w(s))\,\mathrm{d}s = \int_0^t V(w(s))\,\mathrm{d}s \tag{B.16}$$

for all $x = w(0)$ almost surely. Since $|V_n^m| \leq |V| \in \mathcal{K}_\pm(\mathbb{R}^d)$ this is achieved by applying dominated convergence with the help of Remark 2.8.*ii)*. ∎

Proof of Lemma B.1

Since U_n^Λ and V are bounded one has from (2.23)

$$\mathcal{Q}\left(h_{\mathbb{R}^d}^{A,\,\mu U_n^\Lambda+V}\right) = \mathcal{Q}\left(h_{\mathbb{R}^d}^{A,0}\right) \supset \mathcal{Q}\left(h_{\mathbb{R}^d}^{A,\,\mu U_\infty^\Lambda+V}\right) = \mathcal{Q}\left(h_{\mathbb{R}^d}^{A,U_\infty^\Lambda}\right) \qquad (\text{B}.17)$$

and

$$\mathcal{Q}\left(h_{\mathbb{R}^d}^{A,U_\infty^\Lambda}\right) = \left\{\psi \in \mathcal{Q}\left(h_{\mathbb{R}^d}^{A,0}\right) : \sup_{n\in\mathbb{N}} h_{\mathbb{R}^d}^{A,U_n^\Lambda}(\psi,\psi) < \infty\right\}. \qquad (\text{B}.18)$$

Furthermore, we have by construction the monotonicity and boundedness

$$h_{\mathbb{R}^d}^{A,\,\mu U_n^\Lambda+V} \le h_{\mathbb{R}^d}^{A,\,\mu U_{n+1}^\Lambda+V} \le h_{\mathbb{R}^d}^{A,\,\mu U_\infty^\Lambda+V}, \quad n\in\mathbb{N}, \qquad (\text{B}.19)$$

as well as the pointwise convergence

$$\lim_{n\to\infty} h_{\mathbb{R}^d}^{A,\,\mu U_n^\Lambda+V}(\phi,\psi) = h_{\mathbb{R}^d}^{A,\,\mu U_\infty^\Lambda+V}(\phi,\psi), \quad \phi,\psi \in \mathcal{Q}\left(h_{\mathbb{R}^d}^{A,U_\infty^\Lambda}\right). \qquad (\text{B}.20)$$

These facts ensure that the monotone-convergence theorem for not necessarily densely defined sesquilinear forms [55, Theorem 4.1], [69, Theorem 3.1.b)] is applicable and yields the generalized strong resolvent convergence

$$\lim_{n\to\infty}\left\|\left(\frac{1}{1 + H_{\mathbb{R}^d}(A,\mu U_n^\Lambda + V)} - \frac{1}{1 + H_{\mathbb{R}^d}(A,\mu U_\infty^\Lambda + V)}\right)\psi\right\|_2 = 0 \qquad (\text{B}.21)$$

for all $\psi \in \mathrm{L}^2(\mathbb{R}^d)$. This implies [55, Theorem 4.2] the strong convergence of $\left\{f\left(H_{\mathbb{R}^d}(A,\mu U_n^\Lambda + V)\right)\right\}_{n\in\mathbb{N}}$ for all functions f of type (B.3). Especially, (B.5) follows. ∎

Proof of Lemma B.2

The Feynman-Kac-Itô formula (2.34) for $\Lambda = \mathbb{R}^d$ tells

$$\left(\mathrm{e}^{-tH_{\mathbb{R}^d}(A,\mu U_n^\Lambda+V)}\psi\right)(x) = \mathbb{E}_x\left[\mathrm{e}^{-S_t(A,\mu U_n^\Lambda+V|w)}\,\psi(w(t))\right] \qquad (\text{B}.22)$$

for almost all $x \in \mathbb{R}^d$. From Lemma B.1 we learn that by passing to a suitable subsequence the left-hand side of (B.22) tends to the left-hand side of (B.6) as $n \to \infty$ for almost all $x \in \mathbb{R}^d$. For the proof of the convergence of the right-hand side of (B.22) to the desired limit we only have to check

$$\mathrm{e}^{-S_t(0,\mu U_n^\Lambda|w)} \longrightarrow \mathrm{e}^{-S_t(0,\mu U_\infty^\Lambda|w)}\, \Xi_{\Lambda,t}(w) \qquad (\text{B}.23)$$

as $n \to \infty$ for almost all w, because

$$\left|\mathrm{e}^{-S_t(A,\mu U_n^\Lambda+V|w)}\right| \le 1 \qquad (\text{B}.24)$$

allows the application of the dominated-convergence theorem.

If $x = w(0) \notin \overline{\Lambda}$ the left-hand side of (B.23) tends to 0 for all continuous w. Therefore we may assume $x \in \overline{\Lambda}$. In addition it is sufficient to consider a path w which is Hölder

continuous of order $\frac{1}{3}$ [56, Theorem 5.2]. Assume that w leaves Λ and set $0 \le \tau \le t$ to be the first exit time of w from Λ, that is

$$\tau := \inf\{0 < s \le t : w(s) \notin \Lambda\}. \tag{B.25}$$

Then by (B.1), (B.2)

$$\begin{aligned} S_t(0, \mu U_n^\Lambda | w) &\ge \mu \int_0^t \inf\big\{|w(s) - w(\tau)|^{-3}, n\big\} \, \mathrm{d}s \\ &\ge \mu \int_0^t \inf\Big\{\frac{c}{|s - \tau|}, n\Big\} \, \mathrm{d}s \to \infty \end{aligned} \tag{B.26}$$

as $n \to \infty$, where the constant $c > 0$ depends on the path w. Thus (B.23) holds for almost all paths w that do not stay in Λ. But for a path w staying in Λ (B.23) is a consequence of the monotone-convergence theorem. Whence (B.23) is true for almost all w. ∎

Proof of Lemma B.3

By definition, $\mathcal{Q}\big(h_\Lambda^{A,0}\big)$ is the completion of $\mathcal{C}_0^\infty(\Lambda)$ with respect to $\|\bullet\|_{h_{\mathbb{R}^d}^{A,0}}$. Moreover $\mathcal{C}_0^\infty(\Lambda) \subset \mathcal{Q}\big(h_{\mathbb{R}^d}^{A,U_\infty^\Lambda}\big)$. Therefore, $\mathcal{Q}\big(h_\Lambda^{A,0}\big)$ is a subset of the completion of $\mathcal{Q}\big(h_{\mathbb{R}^d}^{A,U_\infty^\Lambda}\big)$. To prove the converse inclusion it is enough to establish that $\mathcal{C}_0^\infty(\Lambda)$ is dense in $\mathcal{Q}\big(h_{\mathbb{R}^d}^{A,U_\infty^\Lambda}\big)$ with respect to $\|\bullet\|_{h_{\mathbb{R}^d}^{A,0}}$. This in turn follows, confer the proof of [13, Theorem 1.13], from the following three assertions:

1) $\mathrm{L}^\infty(\Lambda) \cap \mathcal{Q}\big(h_{\mathbb{R}^d}^{A,U_\infty^\Lambda}\big)$ is dense in $\mathcal{Q}\big(h_{\mathbb{R}^d}^{A,U_\infty^\Lambda}\big)$.

2) $\mathrm{L}_0^\infty(\Lambda) \cap \mathcal{Q}\big(h_{\mathbb{R}^d}^{A,U_\infty^\Lambda}\big)$ is dense in $\mathrm{L}^\infty(\Lambda) \cap \mathcal{Q}\big(h_{\mathbb{R}^d}^{A,U_\infty^\Lambda}\big)$.

3) $\mathcal{C}_0^\infty(\Lambda)$ is dense in $\mathrm{L}_0^\infty(\Lambda) \cap \mathcal{Q}\big(h_{\mathbb{R}^d}^{A,U_\infty^\Lambda}\big)$.

Here denseness is always meant with respect to the form norm $\|\bullet\|_{h_{\mathbb{R}^d}^{A,0}}$ and $\mathrm{L}_0^\infty(\Lambda)$ denotes the space of bounded complex-valued functions with compact support inside Λ.

As to assertion 1)

Fix $t > 0$. Then $\mathrm{e}^{-t H_{\mathbb{R}^d}(A,U_\infty^\Lambda)} \mathrm{L}^2(\Lambda)$ is an operator core for $H_{\mathbb{R}^d}(A, U_\infty^\Lambda)$ by the spectral theorem [51, Section VIII.8]. Consequently it is also a form core for $h_{\mathbb{R}^d}^{A,U_\infty^\Lambda}$. From (B.7) we conclude that $\mathrm{e}^{-t H_{\mathbb{R}^d}(A,U_\infty^\Lambda)} \mathrm{L}^2(\Lambda) \subseteq \mathrm{L}^\infty(\Lambda)$ and therefore

$$\mathrm{L}^\infty(\Lambda) \cap \mathrm{e}^{-t H_{\mathbb{R}^d}(A,U_\infty^\Lambda)} \mathrm{L}^2(\Lambda) \subseteq \mathrm{L}^\infty(\Lambda) \cap \mathcal{Q}\big(h_{\mathbb{R}^d}^{A,U_\infty^\Lambda}\big) \tag{B.27}$$

is dense in $\mathcal{Q}\big(h_{\mathbb{R}^d}^{A,U_\infty^\Lambda}\big)$ with respect to $\|\bullet\|_{h_{\mathbb{R}^d}^{A,U_\infty^\Lambda}}$. Since $\|\bullet\|_{h_{\mathbb{R}^d}^{A,0}} \le \|\bullet\|_{h_{\mathbb{R}^d}^{A,U_\infty^\Lambda}}$ the assertion follows.

As to assertion 2)

Pick $\psi \in \mathrm{L}^\infty(\Lambda) \cap \mathcal{Q}\left(h_{\mathbb{R}^d}^{A,U_\infty^\Lambda}\right)$, then $\vartheta_l \psi \in \mathrm{L}_0^\infty(\Lambda) \cap \mathcal{Q}\left(h_{\mathbb{R}^d}^{A,U_\infty^\Lambda}\right)$ for all $l \in \mathbb{N}$. We want to show $\|\psi - \vartheta_l \psi\|_{h_{\mathbb{R}^d}^{A,0}} \to 0$ as $l \to \infty$. By the construction of ϑ_l we know that $\|(1-\vartheta_l)\phi\|_2 \to 0$ for all $\phi \in \mathrm{L}^2(\Lambda)$. Therefore, it remains to show

$$\lim_{l\to\infty} \|(-i\nabla - A)(\psi - \vartheta_l \psi)\|_2 = 0. \tag{B.28}$$

The triangle inequality gives

$$\|(-i\nabla - A)(\psi - \vartheta_l \psi)\|_2 \leq \|(1 - \vartheta_l)(-i\nabla - A)\psi\|_2 + \|\psi \nabla \vartheta_l\|_2. \tag{B.29}$$

The first term on the right-hand side tends to 0 as $l \to \infty$, because $(-i\nabla - A)\psi \in (\mathrm{L}^2(\Lambda))^d$ due to $\psi \in \mathcal{Q}\left(h_{\mathbb{R}^d}^{A,0}\right)$. We even know $\psi \in \mathcal{Q}\left(h_{\mathbb{R}^d}^{A,U_\infty^\Lambda}\right)$ and therefore

$$\sum_{l\in\mathbb{N}} \|\psi \nabla \vartheta_l\|_2^2 \leq h_{\mathbb{R}^d}^{A,U_\infty^\Lambda}(\psi, \psi) < \infty \tag{B.30}$$

which implies $\|\psi \nabla \vartheta_l\|_2 \to 0$ as $l \to \infty$.

As to assertion 3)

Pick $\psi \in \mathrm{L}_0^\infty(\Lambda) \cap \mathcal{Q}\left(h_{\mathbb{R}^d}^{A,U_\infty^\Lambda}\right)$, then $\delta_r * \psi \in \mathcal{C}_0^\infty(\Lambda)$ for suitably small $r > 0$, where δ_r is the approximate δ-function defined in Appendix A. Since $\lim_{r\downarrow 0} \|\phi - \delta_r * \phi\|_2 = 0$ for all $\phi \in \mathrm{L}^2(\Lambda)$ we aim to show

$$\|(-i\nabla - A)(\psi - \delta_r * \psi)\|_2 \to 0 \tag{B.31}$$

as $r \downarrow 0$ at least for a suitably chosen subsequence.

We first claim that $\nabla\psi \in (\mathrm{L}^2(\Lambda))^d$. This follows from the fact that on the one hand $A\psi \in (\mathrm{L}^2(\Lambda))^d$ because $A \in (\mathrm{L}_{\mathrm{loc}}^2(\Lambda))^d \supset \mathcal{H}_{\mathrm{loc}}(\mathbb{R}^d)$ and $\psi \in \mathrm{L}_0^\infty(\Lambda)$ and on the other hand $(-i\nabla - A)\psi \in (\mathrm{L}^2(\Lambda))^d$ because $\psi \in \mathcal{Q}\left(h_{\mathbb{R}^d}^{A,0}\right)$. Therefore

$$\|(-i\nabla)(\psi - \delta_r * \psi)\|_2 = \|\nabla\psi - \delta_r * (\nabla\psi)\|_2 \to 0 \tag{B.32}$$

as $r \downarrow 0$.

Let $K \subset \Lambda$ denote the compact support of $\delta_R * |\psi|$ for some small $R > 0$. Then for all $0 < r \leq R$

$$|A(x)(\psi - \delta_r * \psi)(x)| \leq 2\|\psi\|_\infty \chi_K(x)\,|A(x)|. \tag{B.33}$$

The right-hand side of this estimate – considered as a function of x – is in $\mathrm{L}^2(\Lambda)$ because $A \in (\mathrm{L}_{\mathrm{loc}}^2(\Lambda))^d$. Moreover, we may choose a subsequence such that $(\psi - \delta_r * \psi)(x) \to 0$ for almost all $x \in \Lambda$ and then get from the dominated-convergence theorem

$$\|A(\psi - \delta_r * \psi)\|_2 \to 0 \tag{B.34}$$

for this subsequence.

The convergences (B.32) and (B.34) imply (B.31) and hence assertion 3. ∎

Proof of Lemma B.4

Recall that

$$
\mathcal{C}_0^\infty(\Lambda) \subset \mathcal{Q}\left(h_{\mathbb{R}^d}^{A,\mu U_\infty^\Lambda+V}\right) = \mathcal{Q}\left(h_{\mathbb{R}^d}^{A,U_\infty^\Lambda}\right) \subseteq \mathcal{Q}\left(h_\Lambda^{A,0}\right) = \mathcal{Q}\left(h_\Lambda^{A,V}\right) \subset \mathrm{L}^2(\Lambda) \tag{B.35}
$$

independent of $\mu > 0$, where the inclusion follows from Lemma B.3. Therefore the monotonicity statement

$$
h_{\mathbb{R}^d}^{A,\mu'U_\infty^\Lambda+V}(\psi,\psi) \geq h_{\mathbb{R}^d}^{A,\mu U_\infty^\Lambda+V}(\psi,\psi) \geq h_\Lambda^{A,V}(\psi,\psi), \quad \mu' > \mu, \tag{B.36}
$$

and the convergence

$$
\lim_{\mu\downarrow 0} h_{\mathbb{R}^d}^{A,\mu U_\infty^\Lambda+V}(\psi,\psi) = h_\Lambda^{A,V}(\psi,\psi) \tag{B.37}
$$

make sense for all $\psi \in \mathcal{Q}\left(h_{\mathbb{R}^d}^{A,U_\infty^\Lambda}\right)$ and follow from the construction of the forms. We conclude with the help of Lemma B.3 that the monotone-convergence theorem for densely defined forms [55, Theorem 3.2], [51, Theorem S.16], see also [69, Theorem 3.1.a)], is applicable and yields the strong resolvent convergence of $H_{\mathbb{R}^d}(A,\mu U_\infty^\Lambda+V)$ to $H_\Lambda(A,V)$ as $\mu \downarrow 0$. According to [51, Theorem VIII.20] this implies (B.8). ∎

Proof of Lemma B.6

First we note that $V_n^m \in \mathrm{L}^\infty(\mathbb{R}^d)$ and $V_\infty^m \in \mathcal{K}(\mathbb{R}^d)$ implies

$$
\mathcal{Q}\left(h_\Lambda^{A,V_n^m}\right) = \mathcal{Q}\left(h_\Lambda^{A,0}\right) = \mathcal{Q}\left(h_\Lambda^{A,V_\infty^m}\right). \tag{B.38}
$$

Moreover, we know that h_Λ^{A,V_∞^m} is bounded from below.

By construction of V_n^m we have the monotonicity

$$
h_\Lambda^{A,V_n^m}(\psi,\psi) \geq h_\Lambda^{A,V_{n+1}^m}(\psi,\psi) \geq h_\Lambda^{A,V_\infty^m}(\psi,\psi) \tag{B.39}
$$

and

$$
h_\Lambda^{A,V_\infty^m}(\psi,\psi) = \lim_{n\to\infty} h_\Lambda^{A,V_n^m}(\psi,\psi) \tag{B.40}
$$

for all $\psi \in \mathcal{Q}\left(h_\Lambda^{A,0}\right)$. These facts imply according to [55, Theorem 3.2], [51, Theorem S.16] the strong resolvent convergence of $H_\Lambda(A,V_n^m)$ to $H_\Lambda(A,V_\infty^m)$ as $n \to \infty$. By [51, Theorem VIII.20] Equation (B.10) follows.

To prove (B.11) we first note that

$$
\Upsilon := \left\{ \psi \in \mathcal{Q}\left(h_\Lambda^{A,0}\right) : \sup_{m\in\mathbb{N}} h_\Lambda^{A,V_\infty^m}(\psi,\psi) < \infty \right\} \tag{B.41}
$$

is dense in $\mathrm{L}^2(\Lambda)$ because $\mathcal{C}_0^\infty(\Lambda) \subset \Upsilon$. Furthermore we have the monotonicity

$$
h_\Lambda^{A,V_\infty^m}(\psi,\psi) \leq h_\Lambda^{A,V_\infty^{m+1}}(\psi,\psi) \leq h_\Lambda^{A,V}(\psi,\psi) \tag{B.42}
$$

and

$$
h_\Lambda^{A,V}(\psi,\psi) = \lim_{m\to\infty} h_\Lambda^{A,V_\infty^m}(\psi,\psi) \tag{B.43}
$$

for all $\psi \in \Upsilon$. With the help of [55, Theorem 3.1], [51, Theorem S.14] we conclude that $\Upsilon = \mathcal{Q}\left(h_\Lambda^{A,V}\right) = \mathcal{Q}\left(h_\Lambda^{A,V^+}\right)$ and the strong resolvent convergence of $H_\Lambda(A,V_\infty^m)$ to $H_\Lambda(A,V)$ as $m \to \infty$. Thus (B.11) follows by using [51, Theorem VIII.20] again. ∎

Appendix C   Brownian-motion estimates

In this appendix we have gathered the probabilistic estimates which play a key rôle in the proofs of Theorems 4.1, 4.3, 4.4, 5.1, 5.5, 6.1, 6.3 and 6.5. Our proofs of these estimates are inspired by ideas for zero vector potentials in [10, Section III], [58, § B.1, § B.10] and the more recent monograph [11, Section 3.2].

Lemma C.1

Let $\{V_n\}_{n\in\mathbb{N}} \subset \mathcal{K}_{\pm}(\mathbb{R}^d)$ obey (5.7). Then

$$\sup_{\tau_1 \leq t \leq \tau_2} \sup_{n\in\mathbb{N}} \left\| \mathrm{e}^{-tH_{\mathbb{R}^d}(0,V_n)} \right\|_{p,q} < \infty \tag{C.1}$$

for all $0 < \tau_1 \leq \tau_2 < \infty$ and $1 \leq p \leq q \leq \infty$. Furthermore, for $p = q$ one may allow $\tau_1 = 0$.

Proof:

The Riesz-Thorin interpolation theorem [50, Theorem IX.17] and the self-adjointness of the semigroup imply that it is enough to prove the Lemma for the two cases $p = q = \infty$, $\tau_1 = 0$ and $p = 1$, $q = \infty$, $\tau_1 > 0$.

In order to show this we first observe that

$$\left\| \mathrm{e}^{-2tH_{\mathbb{R}^d}(0,V_n)} \right\|_{1,\infty} \leq \left(\left\| \mathrm{e}^{-tH_{\mathbb{R}^d}(0,V_n)} \right\|_{2,\infty} \right)^2 \tag{C.2}$$

holds by the semigroup property, the inequality (5.18) and by self-adjointness. Moreover, the inequality

$$\left\| \mathrm{e}^{-tH_{\mathbb{R}^d}(0,V_n)} \right\|_{2,\infty} \leq (2\pi t)^{-d/4} \left(\left\| \mathrm{e}^{-tH_{\mathbb{R}^d}(0,2V_n)} \right\|_{\infty,\infty} \right)^{1/2} \tag{C.3}$$

follows from employing the Cauchy-Schwarz inequality in the Feynman-Kac-Itô formula and from the elementary estimate

$$\left| \left(\mathrm{e}^{-tH_{\mathbb{R}^d}(0,0)} |\psi|^2 \right)(x) \right| \leq (2\pi t)^{-d/2} \left(\|\psi\|_2 \right)^2. \tag{C.4}$$

The combination of (C.2) and (C.3) shows that we are left to treat the case $p = q = \infty$, $\tau_1 = 0$.

The semigroup property (2.37) and the inequality (5.18) yield

$$\left\| \mathrm{e}^{-tH_{\mathbb{R}^d}(0,V_n)} \right\|_{\infty,\infty} \leq \left(\left\| \mathrm{e}^{-tH_{\mathbb{R}^d}(0,V_n)/N} \right\|_{\infty,\infty} \right)^N, \quad N \in \mathbb{N}. \tag{C.5}$$

Therefore it suffices to prove the case $p = q = \infty$, $\tau_1 = 0$ for some small $\tau_2 > 0$. By (5.7) and Remark 5.2.*i)* we may assume τ_2 small enough to ensure

$$\alpha := \sup_{n\in\mathbb{N}} \sup_{x\in\mathbb{R}^d} \mathbb{E}_x \left[\int_0^{\tau_2} V_n^-(w(s))\mathrm{d}s \right] < 1. \tag{C.6}$$

Employing Khas'minskii's lemma [38], [11, Lemma 3.7] we get

$$\mathbb{E}_x\left[\,\mathrm{e}^{-S_t(0,V_n|w)}\,\right] \leq \frac{1}{1-\alpha} \tag{C.7}$$

for all $0 \leq t \leq \tau_2$. By the Feynman-Kac-Itô formula it follows that

$$\left\|\,\mathrm{e}^{-tH_{\mathbb{R}^d}(0,V_n)}\,\right\|_{\infty,\infty} \leq \frac{1}{1-\alpha} \tag{C.8}$$

for all $0 \leq t \leq \tau_2$ and $n \in \mathbb{N}$. ∎

A glance at the preceding proof reveals that (C.7) not only holds for almost all $x \in \mathbb{R}^d$ but for all $x \in \mathbb{R}^d$. We state a frequently used consequence of this separately, see, for example, [11, Proposition 3.8].

Lemma C.2

Let $V \in \mathcal{K}_\pm(\mathbb{R}^d)$. Then

$$\sup_{0\leq t\leq\tau}\,\sup_{x\in\mathbb{R}^d}\,\mathbb{E}_x\left[\,\mathrm{e}^{-S_t(0,V|w)}\,\right] < \infty \tag{C.9}$$

for all $\tau > 0$.

For our purposes it is essential to extend a previously known result for $A = 0$ [11, Proposition 3.9] to $A \neq 0$.

Lemma C.3

Let $A \in \mathcal{H}(\mathbb{R}^d)$ and $V \in \mathcal{K}(\mathbb{R}^d)$. Then

$$\lim_{t\downarrow 0}\,\sup_{x\in\mathbb{R}^d}\,\mathbb{E}_x\left[\,\left|1 - \mathrm{e}^{-S_t(A,V|w)}\right|^p\,\right] = 0 \tag{C.10}$$

for all $0 < p < \infty$.

Lemma C.4

Let $A \in \mathcal{H}(\mathbb{R}^d)$, $\{A_m\}_{m\in\mathbb{N}} \subset \mathcal{H}(\mathbb{R}^d)$, $V \in \mathcal{K}(\mathbb{R}^d)$ and $\{V_n\}_{n\in\mathbb{N}} \subset \mathcal{K}(\mathbb{R}^d)$ such that

$$\lim_{m\to\infty}\left\|\,(A-A_m)^2\,\right\|_{\mathcal{K}(\mathbb{R}^d)} = 0, \tag{C.11}$$

$$\lim_{m\to\infty}\left\|\,\nabla\cdot A - \nabla\cdot A_m\,\right\|_{\mathcal{K}(\mathbb{R}^d)} = 0 \tag{C.12}$$

and

$$\lim_{n\to\infty}\left\|\,V - V_n\,\right\|_{\mathcal{K}(\mathbb{R}^d)} = 0. \tag{C.13}$$

Then

$$\lim_{m,n\to\infty}\,\sup_{0\leq t\leq\tau}\,\sup_{x\in\mathbb{R}^d}\,\mathbb{E}_x\left[\,\left|\mathrm{e}^{-S_t(A,V|w)} - \mathrm{e}^{-S_t(A_m,V_n|w)}\right|^p\,\right] = 0 \tag{C.14}$$

for all $0 \leq \tau < \infty$ and $0 < p < \infty$.

Proof of Lemmas C.3 and C.4:

We start from the inequality

$$\left| e^z - e^{z'} \right| \leq 2^{1+1/q} \, |z - z'|^{1/q} \, e^{\sup\{\mathrm{Re}z, \mathrm{Re}z'\}} \tag{C.15}$$

valid for all $z, z' \in \mathbb{C}$ and $1 \leq q \leq \infty$. It can be inferred from the elementary inequalities $1 - 2\,|u|^{2/q} \leq \cos u$ and $|e^u - 1| \leq |u|^{1/q} \, e^{\sup\{u,0\}}$ for all $u \in \mathbb{R}$.

Choosing q such that $\gamma := pq/(q-p) > 0$ we get from (C.15)

$$2^{-2p} \, \mathbb{E}_x \left[\left| e^{-S_t(A,V|w)} - e^{-S_t(A_m,V_n|w)} \right|^p \right]$$

$$\leq \mathbb{E}_x \left[|S_t(A - A_m, V - V_n|w)|^{p/q} \, e^{-p S_t(0, -V^- - V_n^-|w)} \right]$$

$$\leq \left(\mathbb{E}_x [\, |S_t(A - A_m, V - V_n|w)| \,] \right)^{p/q} \, \left(\mathbb{E}_x \left[e^{-\gamma S_t(0, -V^- - V_n^-|w)} \right] \right)^{p/\gamma} \tag{C.16}$$

$$\leq \left(\mathbb{E}_x [\, |S_t(A - A_m, V - V_n|w)| \,] \right)^{p/q}$$

$$\times \left(\left\| e^{-\tau H_{\mathbb{R}^d}(0, -2\gamma V^-)} \right\|_{\infty,\infty} \sup_{n \in \mathbb{N}} \left\| e^{-\tau H_{\mathbb{R}^d}(0, -2\gamma V_n^-)} \right\|_{\infty,\infty} \right)^{p/2\gamma}.$$

Here we have used the Hölder inequality for the second step and the Cauchy-Schwarz inequality together with the Feynman-Kac-Itô formula for the third step. As remarked similarly in the proof of Theorem 5.5, $V \in \mathcal{K}(\mathbb{R}^d)$, $\{V_n\}_{n \in \mathbb{N}} \subset \mathcal{K}(\mathbb{R}^d)$ and (C.13) imply (5.7). Hence Lemma C.1 can be applied to the right-hand side of (C.16). This guarantees together with (2.40) that it is sufficient to show

$$\lim_{m,n \to \infty} \sup_{0 \leq t \leq \tau} \sup_{x \in \mathbb{R}^d} \mathbb{E}_x [\, |S_t(A - A_m, V - V_n|w)| \,] = 0 \tag{C.17}$$

in order to prove Lemma C.4. Similarly, to verify Lemma C.3 it is enough to show

$$\lim_{t \downarrow 0} \sup_{x \in \mathbb{R}^d} \mathbb{E}_x [\, |S_t(A, V|w)| \,] = 0, \tag{C.18}$$

as can be inferred from (C.16) by putting there $A_m = 0$ and $V_n = 0$ and appealing to (2.40).

To this end, the triangle inequality, the Jensen inequality and the isometry for stochastic integrals [25, Theorem 4.2.5] or [36, Proposition 3.2.10] yield

$$\mathbb{E}_x [\, |S_t(A - A_m, V - V_n|w)| \,]$$

$$\leq \left(\mathbb{E}_x \left[\int_0^t (A_m(w(s)) - A(w(s)))^2 \, ds \right] \right)^{1/2}$$

$$+ \frac{1}{2} \, \mathbb{E}_x \left[\int_0^t |(\nabla \cdot A_m)(w(s)) - (\nabla \cdot A)(w(s))| \, ds \right] \tag{C.19}$$

$$+ \mathbb{E}_x \left[\int_0^t |V_n(w(s)) - V(w(s))| \, ds \right].$$

Now (C.17) is implied by further estimating the right-hand side of (C.19) with the help of

$$\mathbb{E}_x \left[\int_0^t f(w(s)) ds \right] \leq \tau \, \mathbb{E}_x \left[\int_0^1 f\left(\tau^{1/2} w(s)\right) ds \right], \tag{C.20}$$

valid for $f \geq 0$ and $0 \leq t \leq \tau$, and (2.29) in combination with (2.10).

Eventually (C.18) follows directly from (C.19) for $A_m = 0$ and $V_n = 0$ by observing (2.28). ∎

Lemma C.5

Let $A \in \mathcal{H}_{loc}(\mathbb{R}^d)$ and $V \in \mathcal{K}_{\pm}(\mathbb{R}^d)$. Then

$$\lim_{t \downarrow 0} \sup_{x \in K} \mathbb{E}_x \left[\left| 1 - e^{-S_t(A,V|w)} \right|^p \right] = 0 \tag{C.21}$$

for all compact $K \subset \mathbb{R}^d$ and $0 < p < \infty$.

Lemma C.6

Under the hypotheses of Theorem 5.1 one has

$$\lim_{m,n \to \infty} \sup_{0 \leq t \leq \tau} \sup_{x \in K} \mathbb{E}_x \left[\left| e^{-S_t(A,V|w)} - e^{-S_t(A_m,V_n|w)} \right|^p \right] = 0 \tag{C.22}$$

for all compact $K \subset \mathbb{R}^d$, $0 \leq \tau < \infty$ and $0 < p < \infty$.

Proof of Lemmas C.5 and C.6:

By Lemma C.1 and a standard localization technique we will reduce Lemma C.5 and Lemma C.6 to Lemma C.3 and Lemma C.4, respectively. For this purpose it is convenient to define $A_0 := 0$ and $V_0 := 0$. Moreover, we introduce the abbreviation

$$\Upsilon_{p,R}(m,n,t,x) := \mathbb{E}_x \left[\left| e^{-S_t(A,V|w)} - e^{-S_t(A_m,V_n|w)} \right|^p \Xi_{B_R,t}(w) \right]. \tag{C.23}$$

By the Cauchy-Schwarz inequality we obtain

$$\begin{aligned}
\Upsilon_{p,\infty}(m,n,t,x) &\leq \left(\mathbb{E}_x \left[1 - \Xi_{B_R,t}(w) \right] \right)^{1/2} \left(\Upsilon_{2p,\infty}(m,n,t,x) \right)^{1/2} \\
&\quad + \Upsilon_{p,R}(m,n,t,x).
\end{aligned} \tag{C.24}$$

The use of $|z - z'|^{2p} \leq 2^{2p} \left(|z|^{2p} + |z'|^{2p} \right)$ for $z, z' \in \mathbb{C}$ and of the Feynman-Kac-Itô formula leads to

$$\Upsilon_{2p,\infty}(m,n,t,x) \leq 2^{2p} \left(\left\| e^{-\tau H_{\mathbb{R}^d}(0,-2pV^-)} \right\|_{\infty,\infty} + \sup_{n \in \mathbb{N}} \left\| e^{-\tau H_{\mathbb{R}^d}(0,-2pV_n^-)} \right\|_{\infty,\infty} \right). \tag{C.25}$$

Lévy's maximal inequality [56, Equation (7.6′)] gives

$$\lim_{R \to \infty} \sup_{x \in K} \mathbb{E}_x \left[1 - \Xi_{B_R,t}(w) \right] = 0. \tag{C.26}$$

Now we insert the estimate (C.25) into (C.24) and use Lemma C.1 and (C.26) to observe that it is sufficient to show

$$\lim_{t \downarrow 0} \sup_{x \in \mathbb{R}^d} \Upsilon_{p,R}(0,0,t,x) = 0 \tag{C.27}$$

and

$$\lim_{m,n\to\infty} \sup_{0\le t\le\tau} \sup_{x\in\mathbb{R}^d} \Upsilon_{p,R}(m,n,t,x) = 0 \tag{C.28}$$

for all $R > 0$ in order to prove Lemma C.5 and C.6, respectively.

To this end, let Θ_R be as in Appendix A. Then (A.2) implies that $\Upsilon_{p,R}(m,n,t,x)$ as defined in (C.23) does not change its value when one replaces A, A_m, V and V_n by $A\Theta_{R+1}$, $A_m\Theta_{R+1}$, $V\Theta_{R+1}$ and $V_n\Theta_{R+1}$, respectively. This shows that (C.27) follows from Lemma C.3 and (C.28) from Lemma C.4, since $A\Theta_{R+1} \in \mathcal{H}(\mathbb{R}^d)$, $\{A_m\Theta_{R+1}\}_{m\in\mathbb{N}} \subset \mathcal{H}(\mathbb{R}^d)$, $V\Theta_{R+1} \in \mathcal{K}(\mathbb{R}^d)$ and $\{V_n\Theta_{R+1}\}_{n\in\mathbb{N}} \subset \mathcal{H}(\mathbb{R}^d)$ by assumption. ∎

Lemma C.7

If $\Lambda \subseteq \mathbb{R}^d$ is regular, then

$$\lim_{r\downarrow 0} \sup_{|y-x|<r} \mathbb{E}_y[\Xi_{\Lambda,t}(w)] = 0 \tag{C.29}$$

for all $x \in \partial\Lambda$, $t > 0$. If Λ is uniformly regular, then

$$\lim_{r\downarrow 0} \sup_{x\in\partial\Lambda} \sup_{|y-x|<r} \mathbb{E}_y[\Xi_{\Lambda,t}(w)] = 0 \tag{C.30}$$

for all $t > 0$.

Proof:

The first assertion is not new. It is, for example, coded in [6, Corollary II.1.11]. Therefore, we only show the second assertion, but remark that the appropriate simplification of the following proof works for the first one, too.

We start from the simple estimate

$$\mathbb{E}_y[\Xi_{\Lambda,t}(w)] \le \mathbb{E}_y[\Xi_{\Lambda,t-\tau}(w(\bullet+\tau))] \tag{C.31}$$

valid for all $0 < \tau < t$. Using the Markov property of Brownian motion we get

$$\mathbb{E}_y[\Xi_{\Lambda,t-\tau}(w(\bullet+\tau))] = (2\pi\tau)^{-d/2} \int dz\, e^{-(y-z)^2/2\tau}\, \mathbb{E}_z[\Xi_{\Lambda,t-\tau}(w)]. \tag{C.32}$$

This shows that the right-hand side of (C.31), considered as a function of $y \in \mathbb{R}^d$, is uniformly continuous for all $0 < \tau < t$. Therefore,

$$\lim_{r\downarrow 0} \sup_{x\in\partial\Lambda} \sup_{|y-x|<r} \mathbb{E}_y[\Xi_{\Lambda,t}(w)] \le \sup_{x\in\partial\Lambda} \mathbb{E}_x[\Xi_{\Lambda,t-\tau}(w(\bullet+\tau))] \tag{C.33}$$

for all $0 < \tau < t$. Now (C.30) follows from the definition (2.42) of uniform regularity. ∎

Lemma C.8

Let $A \in \mathcal{H}_{loc}(\mathbb{R}^d)$. Then (6.6) and (6.7) hold true for all $t > 0$, $x, y \in \mathbb{R}^d$.

Proof:

For given $t > 0$, $x, y \in \mathbb{R}^d$ we choose $R > 0$ such that $x, y \in B_R$. To show (6.6) we use the estimate

$$
\mathbb{P}_{0,x}^{t,y}\left\{ \int_0^t \left(A(b(s)) \right)^2 \, \mathrm{d}s = \infty \right\}
$$
$$
\leq \mathbb{P}_{0,x}^{t,y}\left\{ \int_0^t \left(\chi_{B_R}(b(s)) \, A(b(s)) \right)^2 \, \mathrm{d}s = \infty \right\} + \mathbb{P}_{0,x}^{t,y}\left\{ \sup_{0 \leq s \leq t} |b(s)| \geq R \right\}. \tag{C.34}
$$

Since

$$
\lim_{R \to \infty} \mathbb{P}_{0,x}^{t,y}\left\{ \sup_{0 \leq s \leq t} |b(s)| \geq R \right\} = 0, \tag{C.35}
$$

it is enough to show that

$$
\mathbb{E}_{0,x}^{t,y}\left[\int_0^t |f(b(s))| \, \mathrm{d}s \right] < \infty \tag{C.36}
$$

for all $f \in \mathcal{K}(\mathbb{R}^d)$. Using (6.5) and the time-reversal symmetry of the Brownian bridge, one has

$$
\mathbb{E}_{0,x}^{t,y}\left[\int_0^t |f(b(s))| \, \mathrm{d}s \right] \leq 2^{1+d/2} \, \mathrm{e}^{(x-y)^2/2t} \sup_{z \in \mathbb{R}^d} \mathbb{E}_z \left[\int_0^{t/2} |f(w(s))| \, \mathrm{d}s \right]. \tag{C.37}
$$

Due to (2.28) the right-hand side is finite and (6.6) is therefore established.

By a reasoning analogous to the one leading to the condition (C.36) it is sufficient to check (6.7) for $A \in \mathcal{H}(\mathbb{R}^d)$ in order to prove it for $A \in \mathcal{H}_{\mathrm{loc}}(\mathbb{R}^d)$. To this end, we observe

$$
2^{-d/2} \, \mathrm{e}^{-(x-y)^2/2t} \left| \mathbb{E}_{0,x}^{t,y}\left[\int_0^t A(b(s)) \cdot \frac{y - b(s)}{t - s} \, \mathrm{d}s \right] \right|
$$
$$
\leq \left| \mathbb{E}_x \left[\int_0^{t/2} A(w(s)) \cdot \frac{y - w(s)}{t - s} \, \mathrm{d}s \right] \right| + \left| \mathbb{E}_y \left[\int_0^{t/2} A(w(s)) \cdot \frac{y - w(s)}{s} \, \mathrm{d}s \right] \right|. \tag{C.38}
$$

Here we have again used (6.5) and the time-reversal symmetry. The first expectation on the right-hand side is seen to be finite for $A^2 \in \mathcal{K}(\mathbb{R}^d)$ by the Cauchy-Schwarz inequality. The second expectation is finite for $\nabla \cdot A \in \mathcal{K}(\mathbb{R}^d)$ due to the partial-integration identity

$$
\mathbb{E}_y \left[A(w(s)) \cdot \frac{y - w(s)}{s} \right] = -\mathbb{E}_y[(\nabla \cdot A)(w(s))]. \tag{C.39}
$$

Hence we are done. ∎

Acknowledgement

References

[1] M. ABRAMOWITZ, I. A. STEGUN (EDS.):
Handbook of mathematical functions
Ninth Printing
New York: Dover 1972

[2] M. AIZENMAN, B. SIMON:
Brownian motion and Harnack inequality for Schrödinger operators
Commun. Pure Appl. Math. **35**, 209–273 (1982)

[3] S. ALBEVERIO, P. BLANCHARD, ZHIMING MA:
Feynman-Kac semigroups in terms of signed smooth measures
Pp. 1–31 in: [33]

[4] S. ALBEVERIO, S. CASATI, U. CATTANEO, D. MERLINI, R. MORESI (EDS.):
Stochastic processes – Physics and geometry
Singapore: World Scientific 1990

[5] J. AVRON, I. HERBST, B. SIMON:
Schrödinger operators with magnetic fields. I. General interactions
Duke Math. J. **45**, 847–883 (1978)

[6] R. F. BASS:
Probabilistic techniques in analysis
New York: Springer 1995

[7] P. BLANCHARD, ZHIMING MA:
Semigroup of Schrödinger operators with potentials given by Radon measures
Pp. 160–195 in: [4]

[8] K. BRODERIX, D. HUNDERTMARK, H. LESCHKE:
Self-averaging, decomposition and asymptotic properties of the density of states for random Schrödinger operators with constant magnetic field
Pp. 98 – 107 in: [30]

[9] K. BRODERIX, D. HUNDERTMARK, W. KIRSCH, H. LESCHKE
The fate of Lifshits tails in magnetic fields
J. Stat. Phys. **80**, 1–22 (1995)

[10] R. CARMONA:
Regularity properties of Schrödinger and Dirichlet semigroups
J. Funct. Anal. **33**, 259–296 (1979)

[11] KAI LAI CHUNG, ZHONGXIN ZHAO:
From Brownian motion to Schrödinger's equation
Berlin: Springer 1995

[12] E. ÇINLAR, K. L. CHUNG, M. J. SHAPE (EDS.):
Seminar on stochastic processes 1992
Boston: Birkhäuser 1993

[13] H. L. Cycon, R. G. Froese, W. Kirsch, B. Simon:
Schrödinger operators, with application to quantum mechanics and global geometry
New York: Springer 1987

[14] M. Demuth, J. A. van Casteren:
On the spectral theory of selfadjoint Feller generators
Rev. Math. Phys. **1**, 325–414 (1989)

[15] M. Demuth, J. A. van Casteren:
Framework and results of stochastic spectral analysis
Pp. 123–132 in: [17]

[16] M. Demuth, J. A. van Casteren:
Stochastic spectral theory for self-adjoint Feller operators
Basel: Birkhäuser 2000

[17] M. Demuth, P. Exner, H. Neidhardt, V. Zagrebnov (Eds.):
Mathematical results in quantum mechanics
Basel: Birkhäuser 1994

[18] M. Demuth, E. Schrohe, B.-W. Schulze (Eds.):
Pseudo-differential calculus and mathematical physics
Berlin: Akademie 1994

[19] M. Demuth, E. Schrohe, B.-W. Schulze, J. Sjöstrand (Eds.):
Schrödinger operators, Markov semigroups, wavelet analysis, operator algebras
Berlin: Akademie 1996

[20] Yu. V. Egorov, M. A. Shubin:
Foundations of the classical theory of partial differential equations
Second Printing
Berlin: Springer 1998

[21] L. Erdős:
Estimates on stochastic oscillatory integrals and on the heat kernel of the magnetic Schrödinger operator
Duke Math. J. **76**, 541–566 (1994)

[22] L. Erdős:
Magnetic Lieb-Thirring inequalities
Commun. Math. Phys. **170**, 629–668 (1995)

[23] L. Erdős:
Dia- and paramagnetism for nonhomogeneous magnetic fields
J. Math. Phys. **38**, 1289–1317 (1997)

[24] L. Erdős:
Lifschitz tail in a magnetic field: the nonclassical regime
Probab. Theory Relat. Fields **112**, 321–371 (1998)

[25] A. FRIEDMANN:
Stochastic differential equations. Volume I
New York: Academic 1975

[26] A. GALINDO, P. PASCUAL:
Quantum mechanics I
Berlin: Springer 1990

[27] A. GALINDO, P. PASCUAL:
Quantum mechanics II
Berlin: Springer 1991

[28] J. GLOVER, M. RAO, R. SONG:
Generalized Schrödinger semigroups
Pp. 143–172 in: [12]

[29] J. GLOVER, M. RAO, H. ŠIKIĆ, R. SONG:
Quadratic forms corresponding to the generalized Schrödinger semigroups
J. Funct. Anal. **125**, 358–378 (1994)

[30] H. GRABERT, A. INOMATA, L. S. SCHULMAN, U. WEISS (EDS.):
Path integrals from meV to MeV: Tutzing '92
Singapore: World Scientific 1993

[31] A. M. HINZ, G. STOLZ:
Polynomial boundedness of eigensolutions and the spectrum of Schrödinger operators
Math. Ann. **294**, 195–211 (1992)

[32] A. M. HINZ:
Regularity of solutions for singular Schrödinger equations
Rev. Math. Phys. **4**, 95–161 (1992)

[33] U. HORNUNG, P. KOTELENEZ, G. PAPANICOLAOU (EDS.):
Random partial differential equations
Basel: Birkhäuser 1991

[34] D. HUNDERTMARK:
Zur Theorie der magnetischen Schrödingerhalbgruppe
Ph.D.-Thesis, Ruhr-Universität Bochum, Bochum 1996

[35] D. HUNDERTMARK:
A second look at the Feynman-Kac-Itô formula
Article in preparation

[36] I. KARATZAS, S. E. SHREVE:
Brownian motion and stochastic calculus
Second Edition
New York: Springer 1991

[37] T. KATO:
      Schrödinger operators with singular potentials
      Israel J. Math. **13**, 135–148 (1972)

[38] R. Z. KHAS'MINSKII:
      Teor. Veroyatnost. i Primenen. **4**, 332–341 (1959)
      English translation:
      On positive solutions of the equation $\mathfrak{A}u + Vu = 0$
      Theory Probab. Appl. **4**, 309–318 (1959)

[39] V. F. KOVALENKO, YU. A. SEMENOV:
      Ukrain. Mat. Zh. **41**, 273–278 (1989)
      English translation:
      L^p-theory of Schrödinger semigroups I
      Ukrainian Math. J. **41**, 246–249 (1989)

[40] H. LEINFELDER, C. G. SIMADER:
      Schrödinger operators with singular magnetic vector potentials
      Math. Z. **176**, 1–19 (1981)

[41] H. LEINFELDER:
      Gauge invariance of Schrödinger operators and related spectral properties
      J. Operator Theory **9**, 163–179 (1983)

[42] V. LISKEVICH, A. MANAVI:
      Dominated semigroups with singular complex potentials
      J. Funct. Anal. **151**, 281–305 (1997)

[43] V. LISKEVICH, YU. A. SEMENOV:
      Some problems on Markov semigroups
      Pp. 163–217 in: [19]

[44] H. MATSUMOTO, N. UEKI:
      Spectral analysis of Schrödinger operators with magnetic fields
      J. Funct. Anal. **140**, 218–255 (1996)

[45] A. MOHAMED, G. D. RAIKOV:
      On the spectral theory of the Schrödinger operator with electromagnetic potential
      Pp. 298–390 in: [18]

[46] S. NAKAO:
      On the spectral distribution of the Schrödinger operator with random potential
      Japan. J. Math. **3**, 111–139 (1977)

[47] M. A. PERELMUTER, YU. A. SEMENOV:
      On decoupling of finite singularities in the scattering theory for the Schrödinger operator with a magnetic field
      J. Math. Phys. **22**, 521–533 (1981)

[48] S. C. Port, C. J. Stone:
      Brownian motion and classical potential theory
      New York: Academic 1978

[49] P. Protter:
      Stochastic integration and differential equations: a new approach
      Third Printing
      Berlin: Springer 1995

[50] M. Reed, B. Simon:
      Methods of modern mathematical physics II: Fourier analysis, self-adjointness
      New York: Academic 1975

[51] M. Reed, B. Simon:
      Methods of modern mathematical physics I: Functional analysis
      Revised and Enlarged Edition
      New York: Academic 1980

[52] D. Revuz, M. Yor:
      Continuous martingales and Brownian motion
      Third Edition
      Berlin: Springer 1999

[53] B. Simon:
      Quantum mechanics for Hamiltonians defined as quadratic forms
      Princeton: Princeton 1971

[54] B. Simon:
      Classical boundary conditions as a technical tool in modern mathematical physics
      Adv. Math. **30**, 268–281 (1978)

[55] B. Simon:
      *A canonical decomposition for quadratic forms with applications to monotone
      convergence theorems*
      J. Funct. Anal. **28**, 377–385 (1978)

[56] B. Simon:
      Functional integration and quantum physics
      New York: Academic 1979

[57] B. Simon:
      Maximal and minimal Schrödinger forms
      J. Operator Theory **1**, 37–47 (1979)

[58] B. Simon:
      Schrödinger semigroups
      Bull. Amer. Math. Soc. (N.S.) **7**, 447–526 (1982)
      Erratum
      Bull. Amer. Math. Soc. (N.S.) **11**, 426 (1984)

[59] L. SMITS:
A remark on smoothing of magnetic Schrödinger semigroups
Commun. Math. Phys. **123**, 527–528 (1989)

[60] L. SMITS:
Moments of certain stochastic integrals occurring in mathematical physics
Proc. Roy. Soc. (Edinburgh) **120A**, 267–282 (1992)

[61] P. STOLLMANN, J. VOIGT:
Perturbation of Dirichlet forms by measures
Potential Analysis **5**, 109–138 (1996)

[62] K.-T. STURM:
Schrödinger semigroups on manifolds
J. Funct. Anal. **118**, 309–350 (1993)

[63] K.-T. STURM:
Schrödinger operators and Feynman-Kac semigroups with arbitrary nonnegative potentials
Expo. Math. **12**, 385–411 (1994)

[64] A.-S. SZNITMAN:
Brownian motion, obstacles and random media
New York: Springer 1998

[65] F. TRÈVES:
Topological vector spaces, distributions and kernels
New York: Academic 1967

[66] H. TRIEBEL:
Higher analysis
Leipzig: Barth 1992

[67] N. UEKI:
On the spectra of random Schrödinger operators with magnetic fields
Osaka J. Math. **31**, 177–187 (1994)

[68] J. VOIGT:
Absorption semigroups, their generators, and Schrödinger semigroups
J. Funct. Anal. **67**, 167–205 (1986)

[69] J. WEIDMANN:
Stetige Abhängigkeit der Eigenwerte und Eigenfunktionen elliptischer Differentialoperatoren vom Gebiet
Math. Scand. **54**, 51–69 (1984)

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